A NUMERICAL CHARACTERIZATION OF NEF ADELIC DIVISORS

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ABSTRACT. To a generically big adelic divisor, we can associate an arithmetic Okounkov body, which is a pair of the geometric Okounkov body and the concave transform of the Green functions. In this paper, we show that the infimum of the concave transform is given by the absolute minimum provided that the divisor is vertically nef. This is a partial generalization of results of Moriwaki (in the curve case) and of Burgos Gil-Moriwaki-Philippon-Sombra (in the toric case).

CONTENTS

1. INTRODUCTION

Let X be a smooth projective variety over a number field K. For each non-Archimedean place $v \in \Sigma_f$, we denote the v-adic completion of K by K_v, and the Berkovich analytic space associated to $X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K_v)$ by X_v^{an}. For $v = \infty$, we set $K_\infty := \mathbb{C}$ and $X_\infty^{\mathrm{an}} := \bigcup_{\iota: K \to \mathbb{C}} (X \times_{\mathrm{Spec}(K),\iota} \mathrm{Spec}(\mathbb{C}))^{\mathrm{an}}$. An *adelic* $\mathbb{R}$-*divisor* $\overline{D} := (D, g)$ on X is a pair of an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on X and a collection of continuous functions, $\sum_{v \in \Sigma_f \cup \{\infty\}} g_v[v]$, where each g_v is a D-Green function on X_v^{an}. An adelic $\mathbb{R}$-divisor $\overline{D}$ is said to be *vertically nef* if D is nef, g_∞ is plurisubharmonic, and every g_P is a uniform limit of nef models. Moreover $\overline{D}$ is said to be *nef* if $\overline{D}$ is vertically nef and $h_{\overline{D}}(x) \geqslant 0$ for every $x \in X(\overline{K})$.

Let $\overline{D}$ be an adelic $\mathbb{R}$-divisor such that D is big. To $\overline{D}$ we can associate a pair $(\Delta(D), G_{\overline{D}})$ called an *arithmetic Okounkov body* of $\overline{D}$, which consists of the geometric Okounkov body $\Delta(D)$ and a upper semicontinuous concave function $G_{\overline{D}}$: $\Delta(D) \to \mathbb{R} \cup \{-\infty\}$. It is known that the arithmetic Okounkov body has rich information on the asymptotic behavior of the number of the global sections with

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small norms (Boucksom and Chen [2], [11]). The most striking are the integral formulas, which state that, if D is big and $\overline{D}$ satisfies some mild condition [2, §3],

$$\widehat{\mathrm{vol}}(\overline{D}) = [K:\mathbb{Q}] \int_{\Delta(D)} \max\{G_{\overline{D}}(u), 0\}\, du$$

and

$$\widehat{\mathrm{vol}}^{\hat{\chi}}(\overline{D}) = [K:\mathbb{Q}] \int_{\Delta(D)} G_{\overline{D}}(u)\, du.$$

Motivated by these formulas, we are interested in the range of the concave function $G_{\overline{D}}$. Our aim is to show that, if D is big and $\overline{D}$ is vertically nef, then the infimum of $G_{\overline{D}}$ is given by the absolute minimum of $\overline{D}$ (Corollary 5.7). In the literature, the following are already known.

- S. Zhang ([16, Corollary (5.7)], [17, Theorem (1.10)]) treated the case where $\overline{D}$ is a vertically nef adelic divisor with D ample.
- If X is a curve and $\overline{D}$ is an integrable adelic $\mathbb{R}$-divisor, then $\overline{D}$ is nef if and only if $\overline{D}$ is pseudo-effective and $\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\deg}(\overline{D}^2)$ (Moriwaki [11, Theorem 7.4.1]).
- If X is a toric variety and $\overline{D}$ is a toric metrized $\mathbb{R}$-divisor [4, §4], then $\overline{D}$ is nef if and only if $\overline{D}$ is vertically nef and $\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\deg}(\overline{D}^{\cdot(\dim X+1)})$ (Burgos Gil, Moriwaki, Philippon, and Sombra [5], [4, Corollary 6.2]).

Given two adelic $\mathbb{R}$-divisors $\overline{D}$ and $\overline{E}$, we write $\overline{D} \sim_{\mathbb{R}} \overline{E}$ if $\overline{D} - \overline{E} = \widehat{(\phi)}$ for a $\phi \in \mathrm{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R}$. If $\overline{D}$ is big, we define the *arithmetic σ-invariant* of $\overline{D}$ at a point $\xi \in X$ as

$$\widehat{\sigma}_\xi(\overline{D}) := \inf\left\{ \mathrm{mult}_\xi(E) \,\middle|\, \overline{D} \sim_{\mathbb{R}} \overline{E} \right\}.$$

Moreover, if $\overline{D}$ is pseudo-effective, we choose a nef and big adelic $\mathbb{R}$-divisor $\overline{A}$ and set $\widehat{\sigma}_\xi(\overline{D}) := \lim_{\varepsilon\to 0+} \widehat{\sigma}_\xi(\overline{D} + \varepsilon\overline{A})$, which does not depend on $\overline{A}$ (see §4 for details). Then, we define the *arithmetic numerical base locus* of $\overline{D}$ as

$$\widehat{\mathrm{NBs}}(\overline{D}) := \left\{ \xi \in X \,\middle|\, \widehat{\sigma}_\xi(\overline{D}) > 0 \right\}.$$

The main results are the following.

Theorem (Theorem 5.6)**.** *Suppose that $\overline{D}$ is vertically nef. Then the following are equivalent.*

(a) *$\overline{D}$ is nef.*
(b) *$G_{\overline{D}} \geqslant 0$.*
(c) *$\widehat{\mathrm{NBs}}(\overline{D}) = \emptyset$.*

Corollary (Corollary 5.7)**.** *Suppose that $\overline{D}$ is vertically nef. Then the infimum of the concave transform of $\overline{D}$ is given by the absolute minimum of $\overline{D}$:*

$$\inf_{u\in\Delta(D)} G_{\overline{D}}(u) = \inf_{x\in X(\overline{K})} h_{\overline{D}}(x).$$

In the curve case [10], [11] or in the toric case [4], one can see that after some blow up there exists a vertically nef adelic $\mathbb{R}$-subdivisor $\overline{Q}(\overline{D})$ of $\overline{D}$ such that $G_{\overline{Q}(\overline{D})} = G_{\overline{D}}$. Thus in these cases $\inf G_{\overline{D}}$ is determined by the above result. One can expect that such picture holds in more general cases. We shall pursue this point elsewhere.

This paper is organized as follows: in §2, we recall some basic properties of the adelic $\mathbb{R}$-divisors ([11]). In §3, we recall the definition and basic properties of

the theory of arithmetic Okounkov bodies by Boucksom-Chen ([2], [11]). In §4, we define the arithmetic σ-invariants of pseudo-effective adelic $\mathbb{R}$-divisors (Definition 4.1), and recall a result of Moriwaki [9] asserting that, if $G_{\overline{D}} \geqslant 0$, then $\widehat{\mathrm{NBs}}(\overline{D}) = \mathrm{NBs}(D)$ holds (Theorem 4.8). In §5, we prove the main results (Theorem 5.6 and Corollary 5.7) and observe some examples.

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2. Adelic $\mathbb{R}$-divisors

Let K be a number field, let O_K be the maximal order of K, and let $\overline{K}$ be an algebraic closure with a fixed $K \subset \overline{K}$. Let Σ_f be the set of all non-Archimedean places of K and let $\Sigma := \Sigma_f \cup \{\infty\}$. For $P \in \Sigma_f$, we denote the P-adic completion of K by K_P, and set

$$|a|_P := \sharp(O_K/P)^{-\operatorname{ord}_P(a)} \tag{2.1}$$

for $a \in K_P$. For $v = \infty$, we set $K_\infty := \mathbb{C}$ and $|\cdot|_\infty$ the usual absolute value on $\mathbb{C}$. Let X be a smooth projective variety of dimension d over K. We denote the Berkovich analytic space associated to $X_P := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K_P)$ by $(X_P^{\mathrm{an}}, \rho_P : X_P^{\mathrm{an}} \to X_P)$ [1, Theorem 3.4.1], and the complex analytic space associated to $X_{\mathbb{C},\mathrm{abs}} := \bigcup_{\iota: K \to \mathbb{C}} X \times_{\mathrm{Spec}(K),\iota} \mathrm{Spec}(\mathbb{C})$ by $(X_\infty^{\mathrm{an}}, \rho_\infty : X_\infty^{\mathrm{an}} \to X_{\mathbb{C},\mathrm{abs}})$. Note that X_v^{an} is a compact Hausdorff space for every $v \in \Sigma$. We set

$$C_P^0(X) := \{\phi : X_P^{\mathrm{an}} \to \mathbb{R} \,|\, \phi \text{ is continuous}\}$$

for $P \in \Sigma_f$ and

$$C_\infty^0(X) := \left\{ \phi : X_\infty^{\mathrm{an}} \to \mathbb{R} \;\middle|\; \begin{array}{l} \phi \text{ is continuous and invariant} \\ \text{under the complex conjugation} \end{array} \right\}.$$

Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor (or an $\mathbb{R}$-divisor for short) on X, which can be written as

$$D = a_1 D_1 + \cdots + a_l D_l \tag{2.2}$$

with prime divisors $D_1, \dots, D_l$ and real numbers $a_1, \dots, a_l$. Since X is smooth, $D_1, \dots, D_l$ are all Cartier and we set $\mathrm{Supp}(D) := \bigcup_{a_i \neq 0} \mathrm{Supp}(D_i)$ endowed with the reduced induced scheme structure. Let $v \in \Sigma$, and let $g_v : (X \setminus \mathrm{Supp}(D))_v^{\mathrm{an}} \to \mathbb{R}$ be a continuous function. We say that g_v is a *D-Green function* if, for every $x \in X_v^{\mathrm{an}}$, there exist an affine open neighborhood $V \subset X$ of $\rho_v(x)$ such that

$$g_v(x') + \sum_{i=1}^{l} a_i \log |f_i(x')|^2 \tag{2.3}$$

extends to a continuous function on V_v^{an}, where f_i is a local defining equation of D_i on V. Note that

$$\begin{aligned}(X \setminus \mathrm{Supp}(D))_v^{\mathrm{an}} &= X_v^{\mathrm{an}} \setminus \mathrm{Supp}(D)_v^{\mathrm{an}} = \rho_v^{-1}(X \setminus \mathrm{Supp}(D)) \\ &= \{x \in X_v^{\mathrm{an}} \,|\, |f_i(x)| \neq 0, \, \forall i\}.\end{aligned}$$

Definition 2.1. Let U be a nonempty open subscheme of $\mathrm{Spec}(O_K)$. A *U-model* of the pair (X, D) is a pair $(\mathscr{X}_U, \mathscr{D}_U)$ having the following properties.

- $\mathscr{X}_U$ is a reduced irreducible normal scheme that is flat and projective over U.

- The generic fiber $\mathscr{X}_U \times_U \operatorname{Spec}(K)$ is K-isomorphic to X.
- Each D_i extends to an effective Cartier divisor $\mathscr{D}_i$ on $\mathscr{X}_U$, and $\mathscr{D}_U$ is an $\mathbb{R}$-divisor on $\mathscr{X}_U$ such that $\mathscr{D}_U \cap X = D$.

In particular, we can write

$$\mathscr{D}_U = a_1\mathscr{D}_1 + \cdots + a_m\mathscr{D}_m \tag{2.4}$$

with vertical effective Cartier divisors $\mathscr{D}_{l+1}, \dots, \mathscr{D}_m$ on $\mathscr{X}_U$ and real numbers $a_{l+1}, \dots, a_m$.

For each $P \in U$, we denote the reduction map by

$$r_P : X_P^{\mathrm{an}} \to \mathscr{X}_P := \mathscr{X}_U \times_U \operatorname{Spec}(O_K/P), \tag{2.5}$$

which sends $x \in X_P^{\mathrm{an}}$ to the closed point $\overline{\{\rho_P(x)\}}^{\mathrm{Zar}} \cap \mathscr{X}_P$ and is known to be anti-continuous [1, Lemma 2.4.1(ii)]. The *D-Green function defined by a model* $(\mathscr{X}_U, \mathscr{D}_U)$ is a continuous function $g_P^{(\mathscr{X}_U,\mathscr{D}_U)} : (X \setminus \operatorname{Supp}(D))_P^{\mathrm{an}} \to \mathbb{R}$ defined as

$$g_P^{(\mathscr{X}_U,\mathscr{D}_U)}(x) := -\sum_{j=1}^{m} a_j \log |f_j(x)|^2,$$

where f_j is a local defining equation of $\mathscr{D}_j$ near $r_P(x)$. Note that $g_P^{(\mathscr{X}_U,\mathscr{D}_U)}$ does not depend on the choice of the expression $\mathscr{D}_U = a_1\mathscr{D}_1 + \cdots + a_m\mathscr{D}_m$ and $f_1, \dots, f_m$.

Definition 2.2. An *adelic $\mathbb{R}$-divisor* on X is a pair $\overline{D} := (D, g_{\overline{D}})$ consisting of an $\mathbb{R}$-divisor D on X and a collection of D-Green functions, $\sum_{v\in\Sigma} g_v[v]$, such that g_∞ is invariant under the complex conjugation and that there exists an open subscheme $U \subset \operatorname{Spec}(O_K)$ and a U-model $(\mathscr{X}_U, \mathscr{D}_U)$ of (X, D) such that $g_P = g_P^{(\mathscr{X}_U,\mathscr{D}_U)}$ for every $P \in U$. We call such an $(\mathscr{X}_U, \mathscr{D}_U)$ a *model of definition for $\overline{D}$*. We denote the $\mathbb{R}$-vector space of all adelic $\mathbb{R}$-divisors on X by $\widehat{\operatorname{Div}}_{\mathbb{R}}^{\mathrm{a}}(X)$.

Let $(\mathscr{X}, \mathscr{D})$ be an O_K-model of (X, D) and let g_∞ be a D-Green function on X_∞^{an} that is invariant under the complex conjugation. Then the arithmetic $\mathbb{R}$-divisor $(\mathscr{D}, g_\infty)$ defines an adelic $\mathbb{R}$-divisor

$$(\mathscr{D}, g_\infty)^{\mathrm{a}} := \left(D, \sum_{v\in\Sigma_f} g_v^{(\mathscr{X},\mathscr{D})}[v] + g_\infty[\infty] \right), \tag{2.6}$$

which we call the *adelic $\mathbb{R}$-divisor corresponding to* $(\mathscr{D}, g_\infty)$.

Let $L := \mathrm{H}^0(X, \mathcal{O}_X)$ and let $K_1 \subset K_2 \subset L$ be subfields of L. For a non-Archimedean place v of K_1, a point $x \in X_v^{\mathrm{an}}$ defines a place w on K_2 lying over v. Thus we have

$$X_v^{\mathrm{an}} = \bigcup_{w|v} X_w^{\mathrm{an}},$$

where w runs over all non-Archimedean places of K_2 lying over v. For a D-Green function g_v on X_v^{an}, the pullback of g_v via $X_w^{\mathrm{an}} \to X_v^{\mathrm{an}}$ is a D-Green function on X_w^{an}. In particular, the notion of adelic $\mathbb{R}$-divisors does not depend on the base field K.

Lemma 2.1. *Let $\overline{D}$ be an adelic $\mathbb{R}$-divisor on X, and let $j : Y \to X$ be a morphism of smooth projective varieties over K such that $j(Y)$ is not contained in* $\operatorname{Supp}(D)$.

(1) *Let* $(\mathscr{X}_U, \mathscr{D}_U)$ *be a model of definition for* $\overline{D}$ *over* $U \subset \mathrm{Spec}(O_K)$, *let* $\mathscr{Y}_U$ *be a normal* U*-model of* Y, *and let* $j_U : \mathscr{Y}_U \to \mathscr{X}_U$ *be a* U*-morphism that extends* j. *Then* $(\mathscr{Y}_U, j_U^*\mathscr{D}_U)$ *is a* U*-model of* (Y, j^*D), *and* $g_v^{(\mathscr{Y}_U, j_U^*\mathscr{D}_U)} = g_v^{(\mathscr{X}_U,\mathscr{D}_U)} \circ j_v^{\mathrm{an}}$.

(2) $j^*\overline{D} := \left(\sum_{i=1}^{l} a_i j^* D_i, \sum_{v\in\Sigma} g_v \circ j_v^{\mathrm{an}}[v]\right)$ *is an adelic* $\mathbb{R}$*-divisor on* Y.

Proof. (1) is nothing but [11, Proposition 2.1.4], and (2) follows from (1). □

A fundamental result on the adelic $\mathbb{R}$-divisors is the following, which we can obtain as a consequence of the Stone-Weierstrass approximation (see [3, Corollary 2.3], [11, Theorem 4.1.3]).

Theorem 2.2. *Let* $\overline{D} = \left(D, \sum_{v\in\Sigma} g_v[v]\right)$ *be an adelic* $\mathbb{R}$*-divisor,* $(\mathscr{X}_U, \mathscr{D}_U)$ *a model of definition for* $\overline{D}$ *over* $U \subset \mathrm{Spec}(O_K)$, *and* $S := \mathrm{Spec}(O_K) \setminus U$. *For* $\varepsilon > 0$, *one can find* O_K*-models* $(\mathscr{X}, \mathscr{D}^1), (\mathscr{X}, \mathscr{D}^2)$ *of* (X, D) *having the following properties.*

- $\mathscr{X}|_U = \mathscr{X} \times_{\mathrm{Spec}(O_K)} U$ *is* U*-isomorphic to* $\mathscr{X}_U$.
- $\mathscr{D}^i|_{\mathscr{X}|_U} = \mathscr{D}_U$ *for* $i = 1, 2$.
- $\overline{D} - \varepsilon\left(0, \sum_{P\in S}[P]\right) \leqslant (\mathscr{D}^1, g_\infty)^{\mathrm{a}} \leqslant \overline{D} \leqslant (\mathscr{D}^2, g_\infty)^{\mathrm{a}} \leqslant \overline{D} + \varepsilon\left(0, \sum_{P\in S}[P]\right)$.

Definition 2.3. Let $\overline{D} = \left(D, \sum_{v\in\Sigma} g_v[v]\right)$ be an adelic $\mathbb{R}$-divisor, let $(\mathscr{X}_U, \mathscr{D}_U)$ be a model of definition for $\overline{D}$ over $U \subset \mathrm{Spec}(O_K)$, and $S := \mathrm{Spec}(O_K) \setminus U$. We say that $\overline{D}$ is *vertically nef* if D is nef, g_∞ is plurisubharmonic, and, for every $\varepsilon > 0$, one can find an O_K-model $(\mathscr{X}, \mathscr{D})$ of (X, D) having the following properties (see [3, Corollary 8.8], [11, Proposition 4.4.2]).

- $\mathscr{X}|_U$ is U-isomorphic to $\mathscr{X}_U$.
- $\mathscr{D}$ is relatively nef and $\mathscr{D}|_{\mathscr{X}_U} = \mathscr{D}_U$.
- $\|g_P^{(\mathscr{X},\mathscr{D})} - g_P\|_{\sup} < \varepsilon$ for every $P \in S$.

Given an adelic $\mathbb{R}$-divisor $\overline{D}$, we set

$$\mathrm{H}^0(X, D) := \{\phi \in \mathrm{Rat}(X)^\times \mid D + (\phi) \geqslant 0\} \cup \{0\}.$$

For $\phi \in \mathrm{H}^0(X, D)$ and for each $x \in X$, $\phi \cdot f_1^{\lfloor a_1 \rfloor} \cdots f_l^{\lfloor a_l \rfloor}$ extends to a regular function near x by Hartogs's extension theorem. Thus

$$|\phi|_{g_v}(x) := |\phi(x)| \exp\left(-\frac{g_v(x)}{2}\right) \tag{2.7}$$

is a continuous function on X_v^{an} taking values in $\mathbb{R}_{\geqslant 0}$. We set $\|\phi\|_{\sup}^{g_v} := \sup_{x\in X_v^{\mathrm{an}}} |\phi|_{g_v}(x)$,

$$\mathrm{H}_f^0(X, \overline{D}) := \{\phi \in \mathrm{H}^0(X, D) \mid \|\phi\|_{\sup}^{g_v} \leqslant 1, \forall\, v \in \Sigma_f\}, \tag{2.8}$$

and

$$\widehat{\mathrm{H}}^0(X, \overline{D}) := \{\phi \in \mathrm{H}_f^0(X, \overline{D}) \mid \|\phi\|_{\sup}^{g_\infty} \leqslant 1\}. \tag{2.9}$$

We say that an adelic $\mathbb{R}$-divisor $\overline{D}$ is *effective* if $\mathrm{mult}_\Gamma(D) \geqslant 0$ for every prime divisor Γ on X and $g_v \geqslant 0$ for every $v \in \Sigma$, and denote by $\overline{D} \geqslant 0$. It is clear that, for $\phi \in \mathrm{Rat}(X)^\times$, $\overline{D} + \widehat{(\phi)}$ is effective if and only if $\phi \in \widehat{\mathrm{H}}^0(X, \overline{D})$.

Let $\overline{D} = \left(D, \sum_{v\in\Sigma} g_v[v]\right), \overline{E} = \left(E, \sum_{v\in\Sigma} h_v[v]\right)$ be adelic $\mathbb{R}$-divisors. Since

$$\|\phi \cdot \psi\|_{\sup}^{g_v+h_v} \leqslant \|\phi\|_{\sup}^{g_v} \cdot \|\psi\|_{\sup}^{h_v}$$

for $\phi \in \mathrm{H}^0(X, D)$, $\psi \in \mathrm{H}^0(X, E)$, and $v \in \Sigma$, we have a natural homomorphism

$$\mathrm{H}_f^0(X, \overline{D}) \otimes \mathrm{H}_f^0(X, \overline{E}) \to \mathrm{H}_f^0(X, \overline{D} + \overline{E}). \tag{2.10}$$

For $t\in\mathbb{R}$, we define $\mathrm{F}^t(X,\overline{D}) := \left\langle \phi\in\mathrm{H}^0_f(X,\overline{D}) \,\middle|\, \|\phi\|_{\sup}^{g_\infty}\leqslant\exp(-t)\right\rangle_{\mathbb{Q}}$ and

$$\mathrm{vol}^t(\overline{D}) := \limsup_{n\to\infty}\frac{\dim_{\mathbb{Q}}\mathrm{F}^{nt}(X,n\overline{D})}{n^{d+1}/(d+1)!}. \tag{2.11}$$

Moreover we set

$$e_{\min}(\overline{D}) := \sup\{t\in\mathbb{R}\mid \dim_{\mathbb{Q}}\mathrm{F}^t(X,\overline{D}) = \dim_{\mathbb{Q}}\mathrm{H}^0(X,D)\},$$
$$e_{\max}(\overline{D}) := \sup\{t\in\mathbb{R}\mid \dim_{\mathbb{Q}}\mathrm{F}^t(X,\overline{D})\geqslant 1\},$$

and $e_{\min}^{\mathrm{asy}}(\overline{D}) := \liminf_{n\to\infty} e_{\min}(n\overline{D})/n$, $e_{\max}^{\mathrm{asy}}(\overline{D}) := \lim_{n\to\infty} e_{\max}(n\overline{D})/n$.

Lemma 2.3. *Let $\overline{D} := \left(D,\sum_{v\in\Sigma}g_v[v]\right)$ be an adelic $\mathbb{R}$-divisor on X.*

(1) *For each $\phi\in\mathrm{H}^0(X,D)$, $\|\phi\|_{\sup}^{g_v}\leqslant 1$ for all but finitely many $v\in\Sigma_f$.*
(2) *$\mathrm{H}^0_f(X,\overline{D})$ is a finitely generated O_K-module.*

In particular, $\mathrm{H}^0_f(X,\overline{D})$ is a finitely generated projective (torsion-free) O_K-module such that the natural inclusion $\mathrm{H}^0_f(X,\overline{D})\subset\mathrm{H}^0(X,D)$ induces $\mathrm{H}^0_f(X,\overline{D})\otimes_{O_K}K = \mathrm{H}^0(X,D)$, and the set $\widehat{\mathrm{H}}^0(X,\overline{D})$ is finite.

Proof. (1) is clear. Since there exists an O_K-model $(\mathscr{X},\mathscr{D})$ of (X,D) such that $\overline{D}\leqslant(\mathscr{D},g_\infty)^{\mathrm{a}}$ (Theorem 2.2), $\mathrm{H}^0_f(X,\overline{D})\subset\mathrm{H}^0(\mathscr{X},\mathscr{D})$ and $\widehat{\mathrm{H}}^0(X,\overline{D})\subset\widehat{\mathrm{H}}^0(\mathscr{X},(\mathscr{D},g_\infty))$. Thus we have (2) (see also [2, §2.2]). □

Given a rational point $x\in X(\overline{K})$, we denote the minimal field of definition for x by $K(x)$. If $x\notin\mathrm{Supp}(D)$, we set

$$\widehat{\deg}_P(\overline{D}|_x) := \frac{1}{2}\sum_{w_j|v}[K(x)_{w_j}:K_v]g(x^{w_j}) \quad\text{and}\quad \widehat{\deg}_\infty(\overline{D}|_x) := \frac{1}{2}\sum_{\iota:K\to\mathbb{C}}g_\infty(x^\iota),$$

where w_j runs over all places of $K(x)$ lying over v and $x^{w_j}\in X_v^{\mathrm{an}}$ is the point defined by $(K(x)_{w_j},w_j)$, and we define the *height* of x by

$$h_{\overline{D}}(x) := \frac{1}{[K(x):\mathbb{Q}]}\sum_{v\in\Sigma}\widehat{\deg}_v(\overline{D}|_x). \tag{2.12}$$

For adelic $\mathbb{R}$-divisors $\overline{D}$ and $\overline{E}$ such that $x\notin\mathrm{Supp}(D)\cup\mathrm{Supp}(E)$, we have $h_{\overline{D}+\overline{E}}(x) = h_{\overline{D}}(x)+h_{\overline{E}}(x)$, and, for $\phi\in\mathrm{Rat}(X)^\times$ such that $x\notin\mathrm{Supp}((\phi))$, $h_{\widehat{(\phi)}}(x)=0$. Given general $x\in X(\overline{K})$, one can find a $\phi\in\mathrm{Rat}(X)^\times\otimes_{\mathbb{Z}}\mathbb{R}$ such that $x\notin\mathrm{Supp}(D+(\phi))$. Then we set

$$h_{\overline{D}}(x) := h_{\overline{D}+\widehat{(\phi)}}(x), \tag{2.13}$$

which does not depend on the choice of ϕ. The following are clear by definition.

Lemma 2.4. *Let $\overline{D},\overline{E}$ be two adelic $\mathbb{R}$-divisors on X, and let $x\in X(\overline{K})$.*

(1) *$h_{\overline{D}+\overline{E}}(x) = h_{\overline{D}}(x)+h_{\overline{E}}(x)$.*
(2) *For any $\phi\in\mathrm{Rat}(X)^\times$, we have $h_{\widehat{(\phi)}}(x)=0$.*
(3) *If $\overline{D}$ is effective and $x\notin\mathrm{Supp}(D)$, then $h_{\overline{D}}(x)\geqslant 0$.*
(4) *Let $Y\subset X$ be a closed subvariety such that Y is not contained in $\mathrm{Supp}(D)$. Then $h_{\overline{D}|_Y}(x) = h_{\overline{D}}(x)$ for $x\in Y(\overline{K})\subset X(\overline{K})$.*

Definition 2.4. Let $\overline{D}$ be an adelic $\mathbb{R}$-divisor on X.

(nef): We say that $\overline{D}$ is *nef* if $\overline{D}$ is vertically nef and $h_{\overline{D}}(x) \geqslant 0$ for every $x \in X(\overline{K})$. We denote the cone of all nef adelic $\mathbb{R}$-divisors on X by $\widehat{\mathrm{Nef}}^{\mathrm{a}}_{\mathbb{R}}(X)$.

(integrable): We say that $\overline{D}$ is *integrable* if $\overline{D}$ can be written as a difference of two nef adelic $\mathbb{R}$-divisors, and denote the $\mathbb{R}$-subspace of all integrable adelic $\mathbb{R}$-divisors on X by $\widehat{\mathrm{Int}}^{\mathrm{a}}_{\mathbb{R}}(X)$. As in [11, §4.5], we can extend the usual arithmetic intersection numbers to obtain the symmetric multilinear map

$$\widehat{\mathrm{Int}}^{\mathrm{a}}_{\mathbb{R}}(X)^{d+1} \to \mathbb{R}, \quad (\overline{D}_0, \dots, \overline{D}_d) \mapsto \widehat{\deg}(\overline{D}_0 \cdots \overline{D}_d).$$

(big): For an adelic divisor $\overline{D}$ on X, we set

$$\widehat{\mathrm{vol}}(\overline{D}) := \limsup_{n\to\infty} \frac{\log \sharp \widehat{\mathrm{H}}^0(X, n\overline{D})}{n^{d+1}/(d+1)!}.$$

We know that the function $\widehat{\mathrm{Div}}^{\mathrm{a}}_{\mathbb{R}}(X) \to \mathbb{R}$, $\overline{D} \mapsto \widehat{\mathrm{vol}}(\overline{D})$, is continuous: that is,

$$\lim_{\substack{\varepsilon_1,\dots,\varepsilon_m\to 0\\ \|\varphi_1\|_{\sup},\dots,\|\varphi_n\|_{\sup}\to 0}} \widehat{\mathrm{vol}}\left(\overline{D} + \sum_{i=1}^m \varepsilon_i \overline{E}_i + \sum_{j=1}^n (0, \varphi_j[v_j])\right) = \widehat{\mathrm{vol}}(\overline{D})$$

for $\overline{D}, \overline{E}_1, \dots, \overline{E}_m \in \widehat{\mathrm{Div}}^{\mathrm{a}}_{\mathbb{R}}(X)$, $v_1, \dots, v_n \in \Sigma$, and $\varphi_j \in C^0_{v_j}(X)$ [11, §5.2]. We say that $\overline{D}$ is *big* if $\widehat{\mathrm{vol}}(\overline{D}) > 0$, and denote the cone of all big adelic $\mathbb{R}$-divisors on X by $\widehat{\mathrm{Big}}^{\mathrm{a}}_{\mathbb{R}}(X)$.

(pseudo-effective): We say that $\overline{D}$ is *pseudo-effective* if, for every big adelic $\mathbb{R}$-divisor $\overline{A}$, we have $\widehat{\mathrm{vol}}(\overline{D} + \overline{A}) > 0$. We denote the cone of all pseudo-effective adelic $\mathbb{R}$-divisors on X by $\widehat{\mathrm{PE}}^{\mathrm{a}}_{\mathbb{R}}(X)$.

For an adelic divisor $\overline{D}$ on X, we set

$$\widehat{\mathrm{vol}}^{\hat{\chi}}(\overline{D}) := \limsup_{n\to\infty} \frac{\hat{\chi}(\mathrm{H}^0(X, nD), (\|\cdot\|^{ng_v}_{\sup})_v)}{n^{d+1}/(d+1)!},$$

where $\hat{\chi}(\mathrm{H}^0(X, nD), (\|\cdot\|^{ng_v}_{\sup})_v)$ is the Euler characteristic of the adelic vector space $(\mathrm{H}^0(X, nD), (\|\cdot\|^{ng_v}_{\sup})_v)$ ([2, (3.1)]). By [15, Theorem 1.2], this actually takes values in $\mathbb{R}$, and obtain the following by standard arguments.

Lemma 2.5. *For $\mathbb{K} = \emptyset, \mathbb{Q}, \mathbb{R}$, we denote the cone of all adelic $\mathbb{K}$-divisors $\overline{D}$ with ample D by $\widehat{\mathrm{GAmp}}^{\mathrm{a}}_{\mathbb{K}}(X)$.*

(1) *For $\overline{A}, \overline{B} \in \widehat{\mathrm{GAmp}}^{\mathrm{a}}(X)$, there exists an integer $m_0 \gg 1$ such that*

$$e_{\min}(m\overline{A} + n\overline{B}) \geqslant e_{\min}(m\overline{A}) + e_{\min}(n\overline{B}).$$

for every $m \geqslant m_0$ and for every $n \geqslant m_0$.

(2) *The function $e^{\mathrm{asy}}_{\min} : \widehat{\mathrm{GAmp}}^{\mathrm{a}}(X) \to \mathbb{R}$ uniquely extends to a continuous function $e^{\mathrm{asy}}_{\min} : \widehat{\mathrm{GAmp}}^{\mathrm{a}}_{\mathbb{R}}(X) \to \mathbb{R}$.*

(3) *The function $\widehat{\mathrm{vol}}^{\hat{\chi}} : \widehat{\mathrm{GAmp}}^{\mathrm{a}}(X) \to \mathbb{R}$ uniquely extends to a continuous function $\widehat{\mathrm{vol}}^{\hat{\chi}} : \widehat{\mathrm{GAmp}}^{\mathrm{a}}_{\mathbb{R}}(X) \to \mathbb{R}$.*

(4) *If D is ample and $\overline{D}$ is vertically nef, then $\widehat{\mathrm{vol}}^{\hat{\chi}}(\overline{D}) = \widehat{\deg}(\overline{D}^{\cdot d+1})$.*

Remark 2.6. As is well-known, for an ample $\mathbb{R}$-divisor D, the section algebra $\bigoplus_{n\geqslant 0}\mathrm{H}^0(X,nD)$ is finitely generated over K if and only if D is an ample $\mathbb{Q}$-divisor ([13, Chap. III, Remark 1.17]).

3. Arithmetic Okounkov bodies

Fix an embedding $K\subset\overline{K}$ and let X be a smooth projective variety of dimension d over K. In this section, we suppose that X is geometrically irreducible over K: that is, $K=\mathrm{H}^0(X,\mathcal{O}_X)$ and $X_{\overline{K}}:=X\times_{\mathrm{Spec}(K)}\mathrm{Spec}(\overline{K})$ is reduced irreducible. A *monomial order* $\leqslant$ on $\mathbb{R}^d$ is a total order on $\mathbb{R}^d$ such that (1) $I\in\mathbb{R}^d_{\geqslant 0}$ implies $I\geqslant 0$, and (2) for every $I_1,I_2,I_3\in\mathbb{R}^d$, $I_1\leqslant I_2$ implies $I_1+I_3\leqslant I_2+I_3$. Let $x\in X_{\overline{K}}$ be a smooth closed point, and let $z_x:=(z_1,\dots,z_d)$ be a local parameter system at x. Given $f\in\mathcal{O}_{X_{\overline{K}},x}$, we expand f as a formal power series $f=\sum_{I\in\mathbb{Z}^d_{\geqslant 0}}a_Iz^I$ in $\widehat{\mathcal{O}}_{X_{\overline{K}},x}=\overline{K}[[z_1,\dots,z_d]]$, and set

$$\boldsymbol{v}_{z_x,\leqslant}(f):=\begin{cases}\min_{\leqslant}\{I\in\mathbb{Z}^d_{\geqslant 0}\,|\,a_I\neq 0\} & \text{if } f\neq 0\\ +\infty & \text{if } f=0,\end{cases}$$

where the minimum is taken with respect to a monomial order $\leqslant$ on $\mathbb{R}^d$. We can easily see that $\boldsymbol{v}_{z_x,\leqslant}$ has the valuation-like properties: that is, for $f,g\in\mathcal{O}_{X_{\overline{K}},x}$,

- $\boldsymbol{v}_{z_x,\leqslant}(f)=+\infty$ if and only if $f=0$,
- $\boldsymbol{v}_{z_x,\leqslant}(f+g)\geqslant\min\{\boldsymbol{v}_{z_x,\leqslant}(f),\boldsymbol{v}_{z_x,\leqslant}(g)\}$, and
- $\boldsymbol{v}_{z_x,\leqslant}(fg)=\boldsymbol{v}_{z_x,\leqslant}(f)+\boldsymbol{v}_{z_x,\leqslant}(g)$.

In particular, we can extend the function $\boldsymbol{v}_{z_x,\leqslant}:\mathcal{O}_{X_{\overline{K}},x}\to\mathbb{Z}^d_{\geqslant 0}\cup\{+\infty\}$ to

$$\boldsymbol{v}_{z_x,\leqslant}:\mathrm{Rat}(X_{\overline{K}})\to\mathbb{Z}^d\cup\{+\infty\}\tag{3.1}$$

by linearity. Let $D:=a_1D_1+\cdots+a_lD_l$ be an $\mathbb{R}$-divisor on $X_{\overline{K}}$, where $D_1,\dots,D_l$ are effective Cartier divisors and $a_1,\dots,a_l\in\mathbb{R}$. We choose local defining equations $f_1,\dots,f_l$ of $D_1,\dots,D_l$ at x, respectively, and define

$$\boldsymbol{v}_{z_x,\leqslant}(D):=\sum_{i=1}^l a_i\boldsymbol{v}_{z_x,\leqslant}(f_i),\tag{3.2}$$

which does not depend on the choice of the expression $D=\sum_{i=1}^l a_iD_i$ and $f_1,\dots,f_l$.

Let $\xi\in X_{\overline{K}}$ be a point such that $\overline{\{\xi\}}$ is defined by $z_1=\cdots=z_h=0$ at x. The *multiplicity* at ξ of a non-zero $f=\sum_I a_Iz^I$ is defined as the order of vanishing at ξ: that is,

$$\mathrm{mult}_\xi(f):=\min\{i_1+\cdots+i_h\,|\,a_{(i_1,\dots,i_d)}\neq 0,\,\exists\, i_{h+1},\dots,i_d\}.\tag{3.3}$$

We set $L_\xi(f):=\sum_{i_1+\cdots+i_h=\mathrm{mult}_\xi(f)}a_Iz^I$.

Lemma 3.1. *Let $D=a_1D_1+\cdots+a_lD_l$ and $f_1,\dots,f_l$ as above. Suppose that D is nonzero and effective.*

1. *for any $\xi\in X_{\overline{K}}$, the function $X\ni\xi\mapsto\mathrm{mult}_\xi(D)\in\mathbb{Z}_{\geqslant 0}$ is upper semicontinuous.*
2. *Let H be a smooth hypersurface that is defined by $z_1=0$ at x. If f_i does not vanish along $\{z_1=0\}$ for all i, then $\mathrm{mult}_x(D\cap H)\geqslant\mathrm{mult}_x(D)$, and, if $L_x(f_i)$ does not vanish along $\{z_1=0\}$ for all i, then $\mathrm{mult}_x(D\cap H)=\mathrm{mult}_x(D)$.*

Proof. (1): First, we assume that D is a divisor, and let $\mathcal{D}^{\leqslant \ell}$ be the ideal sheaf locally generated by derivatives of order $\leqslant \ell$ of local sections of $\mathcal{O}_X(-D)$. Since $\mathrm{mult}_\xi(D) = \min\{\ell \in \mathbb{Z}_{\geqslant 0} \,|\, (\mathcal{O}_{X_{\overline{K}}}/\mathcal{D}^{\leqslant \ell})_\xi = 0\}$, the assertion holds.

Next in general, we can write $D = a_1 D_1 + \cdots + a_l D_l$ with positive $a_1, \dots, a_l$. For every $u \in \mathbb{R}$,

$$\{\xi \in X_{\overline{K}} \,|\, \mathrm{mult}_\xi(D) < u\} = \bigcup_{a_1 u_1 + \cdots + a_l u_l = u} \bigcap_{i=1}^{l} \{\xi \in X_{\overline{K}} \,|\, \mathrm{mult}_\xi(D_i) < u_i\}$$

is Zariski open.

(2): Since $\mathrm{mult}_x(D \cap H) = \min\{i_1 + \cdots + i_d \,|\, a_{(i_1,\dots,i_d)} \neq 0,\, i_1 = 0\}$, we have (2). $\square$

Let $\overline{D}$ be an adelic $\mathbb{R}$-divisor on X such that D is big, and let $\Delta(D)$ be the usual Okounkov body of $D_{\overline{K}}$ ([8], [2]): that is,

$$\Delta(D) := \text{the closed convex hull of } \bigcup_{n \geqslant 1} \frac{1}{n} \boldsymbol{v}_{z_x,\leqslant}(\mathrm{H}^0(X, nD) \setminus \{0\}). \tag{3.4}$$

For $t \in \mathbb{R}$, we define a compact convex body $\Delta^t(\overline{D})$ as

$$\Delta^t(\overline{D}) := \text{the closed convex hull of } \bigcup_{n \geqslant 1} \frac{1}{n} \boldsymbol{v}_{z_x,\leqslant}(\mathrm{F}^{nt}(X, n\overline{D}) \setminus \{0\}). \tag{3.5}$$

Remark 3.2. (1) The above definitions (3.4) and (3.5) are precisely a *translate* of the definitions given by Boucksom-Chen [2] and Moriwaki [11]. Note that

$$\left\{ D + (\phi) \;\middle|\; \phi \in \bigcup_{n \geqslant 1} \frac{1}{n} \left(\mathrm{F}^{nt}(X, n\overline{D}) \setminus \{0\}\right) \right\} = \left\{ E \;\middle|\; \overline{E} \sim_{\mathbb{Q}} \overline{D},\, \overline{E} \geqslant (0, 2t[\infty]) \right\}.$$

Our definitions have a merit that $\overline{D} \leqslant \overline{E}$ implies $\Delta(D) \subset \Delta(E)$ and $\Delta^t(\overline{D}) \subset \Delta^t(\overline{E})$ for $t \in \mathbb{R}$.

(2) For every $\lambda \in \mathbb{R}$, $\Delta^t(\overline{D} - (0, 2\lambda[\infty])) = \Delta^{t+\lambda}(\overline{D})$.

(3) As in [2, §1.3], the sequence of nonempty compact subsets, $\left(\Delta^t(\overline{D})\right)_{t < e^{\mathrm{asy}}_{\max}(\overline{D})}$, is monotone decreasing,

$$\text{the interior of } \Delta(D) = \bigcup_{t \in \mathbb{R}} \text{the interior of } \Delta^t(\overline{D}),$$

and $\{u \in \Delta(D) \,|\, G_{\overline{D}}(u) \geqslant t\} = \bigcap_{t' < t} \Delta^{t'}(\overline{D})$ for every $t \in \mathbb{R}$.

Lemma 3.3. *We have*

$$\Delta(D) = \textit{the closed convex hull of } \{\boldsymbol{v}_{z_x,\leqslant}(E - D) \,|\, E \sim_{\mathbb{R}} D,\, E \geqslant 0\}$$

and

$$\Delta^t(\overline{D}) = \textit{the closed convex hull of } \{\boldsymbol{v}_{z_x,\leqslant}(E - D) \,|\, \overline{E} \sim_{\mathbb{R}} \overline{D},\, \overline{E} \geqslant (0, 2t[\infty])\}$$

for $t \in \mathbb{R}$.

Proof. We prove the latter half. The inclusion $\subset$ is clear. Suppose that $\phi_1, \dots, \phi_l \in \mathrm{Rat}(X)^\times$ and $a_1, \dots, a_l \in \mathbb{R}$ satisfy

$$\overline{E} := \overline{D} + a_1 \widehat{(\phi_1)} + \cdots + a_l \widehat{(\phi_l)} \geqslant (0, 2t[\infty]).$$

By [9], we can write each $\widehat{(\phi_i)}$ as a difference of two effective adelic (in fact, arithmetic) divisors: $\widehat{(\phi_i)} = \overline{A}_i - \overline{B}_i$. We set $\overline{F} := \sum_{i=1}^{l} \overline{B}_i$. Then for any $\varepsilon > 0$, we can approximate $E - D$ by elements in

$$\left\{ b_1(\phi_1) + \cdots + b_l(\phi_l) \;\middle|\; \begin{array}{l} b_1, \ldots, b_l \in \mathbb{Q}, \\ \overline{D} + \varepsilon\overline{F} + b_1\widehat{(\phi_1)} + \cdots + b_l\widehat{(\phi_l)} \geqslant (0, 2t[\infty]) \end{array} \right\}.$$

Thus we have

$$\boldsymbol{v}_{z_x,\leqslant}(E - D) \in \bigcap_{\varepsilon > 0} \Delta^0(\overline{D} + \varepsilon\overline{F} - (0, 2t[\infty])).$$

Since the right-hand-side is a compact convex subset with volume $\mathrm{vol}^t(\overline{D})$ (see Proposition 3.4 (2) below), we have

$$\boldsymbol{v}_{z_x,\leqslant}(E - D) \in \bigcap_{\varepsilon > 0} \Delta^t(\overline{D} + \varepsilon\overline{F}) = \Delta^t(\overline{D}).$$

$\square$

The *concave transform* $G_{\overline{D}} : \Delta(D) \to \mathbb{R} \cup \{-\infty\}$ is a upper semicontinuous concave function defined as

$$G_{\overline{D}}(u) := \sup\{t \in \mathbb{R} \,|\, u \in \Delta^t(\overline{D})\}$$

for $u \in \Delta(D)$. Since $G_{\overline{D}}$ is concave, $G_{\overline{D}}$ is continuous and takes values in $\mathbb{R}$ over the interior of $\Delta(D)$, and can be discontinuous over the boundary of $\Delta(D)$. By the same arguments as in [2], we have

$$\widehat{\mathrm{vol}}(\overline{D}) = (d+1)![K : \mathbb{Q}] \int_{\Delta(D)} \max\{G_{\overline{D}}(u), 0\}\, du \tag{3.6}$$

and, if D is ample, $G_{\overline{D}}$ is bounded and

$$\widehat{\mathrm{vol}}^{\hat{\chi}}(\overline{D}) = (d+1)![K : \mathbb{Q}] \int_{\Delta(D)} G_{\overline{D}}(u)\, du \tag{3.7}$$

(see [2], [11]). Here the integrals are taken with respect to the Lebesgue measure on $\mathbb{R}^d$.

Proposition 3.4. *Suppose that D is big.*

(1) *For any $t \in \mathbb{R}$ with $t < e_{\max}^{\mathrm{asy}}(\overline{D})$, we have*

$$\mathrm{vol}(\Delta^t(\overline{D})) = \frac{\mathrm{vol}^t(\overline{D})}{(d+1)!} = \lim_{p \to \infty} \frac{\dim_K \mathrm{F}^{pt}(X, p\overline{D})}{p^{d+1}},$$

where the first vol *stands for the Lebesgue measure on $\mathbb{R}^d$.*

(2) *The function*

$$\mathrm{vol}^0 : \widehat{\mathrm{Big}}^{\mathrm{a}}_{\mathbb{R}}(X) \to \mathbb{R}, \quad \overline{D} \mapsto \mathrm{vol}^0(\overline{D})^{\frac{1}{d+1}},$$

is positively homogeneous of degree one, concave, and continuous: that is,

$$\mathrm{vol}^0(p\overline{D}) = p^{d+1}\,\mathrm{vol}^0(\overline{D})$$

and

$$\mathrm{vol}^0(\overline{D} + \overline{E})^{\frac{1}{d+1}} \geqslant \mathrm{vol}^0(\overline{D})^{\frac{1}{d+1}} + \mathrm{vol}^0(\overline{E})^{\frac{1}{d+1}}$$

for every $\overline{D},\overline{E}\in\widehat{\mathrm{Big}}^{\mathrm{a}}_{\mathbb{R}}(X)$, *and*

$$\lim_{\substack{\varepsilon_1,\dots,\varepsilon_m\to 0\\ \|\varphi_1\|_{\sup},\dots,\|\varphi_n\|_{\sup}\to 0}}\mathrm{vol}^0\left(\overline{D}+\sum_{i=1}^m\varepsilon_i\overline{E}_i+\sum_{j=1}^n(0,\varphi_j[v_j])\right)=\mathrm{vol}^0(\overline{D})$$

for $\overline{D}\in\widehat{\mathrm{Big}}^{\mathrm{a}}_{\mathbb{R}}(X)$, $\overline{E}_1,\dots,\overline{E}_m\in\widehat{\mathrm{Div}}^{\mathrm{a}}_{\mathbb{R}}(X)$, $v_1,\dots,v_n\in\Sigma$, *and* $\varphi_j\in C^0_{v_j}(X)$.

(3) *For any* $t\in\mathbb{R}$ *with* $t<e^{\mathrm{asy}}_{\max}(\overline{D})$, *we have*

$$\Delta^t(\overline{D})=\overline{\{u\in\Delta(D)\,|\,G_{\overline{D}}(u)>t\}}=\{u\in\Delta(D)\,|\,G_{\overline{D}}(u)\geqslant t\}.$$

Proof. If $t<e^{\mathrm{asy}}_{\max}(\overline{D})$, then $\bigoplus_{n\geqslant 0}\mathrm{F}^{nt}(X,n\overline{D})$ contains an ample series [2, Definition 1.1 and Lemma 1.6]. Thus (1) is nothing but [8, Theorem 2.13]. For (3), we refer to [9, Lemma 3.3].

(2): For adelic $\mathbb{R}$-divisors $\overline{D}$ and $\overline{E}$, we have homomorphisms

$$\mathrm{F}^0(X,n\overline{D})\otimes\mathrm{F}^0(X,n\overline{E})\to\mathrm{F}^0(X,n(\overline{D}+\overline{E}))$$

for $n\geqslant 1$. Thus (2) follows from the same arguments as in [8, Corollary 4.12]. □

Lemma 3.5. *If D is big, then the following two conditions are equivalent.*

(a) $G_{\overline{D}}(u)\geqslant 0$ *for every* $u\in\Delta(D)$.
(b) $\mathrm{vol}^0(\overline{D})=\mathrm{vol}(D)$.

Moreover, if D is ample, then the following is also equivalent to the above.

(c) $\widehat{\mathrm{vol}}(\overline{D})=\mathrm{vol}^{\hat{\chi}}(\overline{D})$.

Moreover, if D is ample and $\overline{D}$ is vertically nef, then the following is also equivalent to the above.

(d) $\widehat{\mathrm{vol}}(\overline{D})=\widehat{\deg}(\overline{D}^{\cdot(d+1)})$.

In particular, if D is big, then

$$\inf_{u\in\Delta(D)}G_{\overline{D}}(u)=\sup\{t\in\mathbb{R}\,|\,\mathrm{vol}^t(\overline{D})=\mathrm{vol}(D)\}.$$

Proof. (a) ⇔ (b): Both are equivalent to the condition that $\Delta^0(\overline{D})=\Delta(D)$. (a) ⇔ (c) follows from the integral formulas (3.6), (3.7), and (c) ⇔ (d) follows from Lemma 2.5 (4).

Set $\lambda_1:=\inf_{u\in\Delta(D)}G_{\overline{D}}(u)$ and $\lambda_2:=\sup\{t\in\mathbb{R}\,|\,\mathrm{vol}^t(\overline{D})=\mathrm{vol}(D)\}$. Since $G_{\overline{D}-(0,2\lambda_1[\infty])}(u)=G_{\overline{D}}(u)-\lambda_1\geqslant 0$, we have $\mathrm{vol}^{\lambda_1}(\overline{D})=\mathrm{vol}^0(\overline{D}-2\lambda_1[\infty])=\mathrm{vol}(D)$ and $\lambda_1\leqslant\lambda_2$. On the other hand, for any $\varepsilon>0$, $\mathrm{vol}^{\lambda_2-\varepsilon}(\overline{D})=\mathrm{vol}^0(\overline{D}-2(\lambda_2-\varepsilon)[\infty])=\mathrm{vol}(D)$. Thus $G_{\overline{D}-2(\lambda_2-\varepsilon)[\infty]}(u)=G_{\overline{D}}(u)-(\lambda_2-\varepsilon)\geqslant 0$ for every $u\in\Delta(D)$. Hence $\lambda_1\geqslant\lambda_2$. □

Corollary 3.6. *If $\overline{D}$ is nef and big, then* $\mathrm{vol}^0(\overline{D})=\mathrm{vol}(D)$.

Remark 3.7. An arithmetic divisor $\overline{\mathscr{H}}=(\mathscr{H},g_{\overline{\mathscr{H}}})$ on $\mathscr{X}$ is said to be *ample* if $g_{\overline{\mathscr{H}}}$ defines a C^∞-Hermitian metric, $\mathscr{H}$ is ample, $\omega(\overline{\mathscr{H}})$ is a positive $(1,1)$-form, and, for every $n\gg 1$, $\mathrm{H}^0(\mathscr{X},n\mathscr{H})$ is generated by sections with supremum norms less than one. Let $\mathbb{K}:=\mathbb{Q}$ or $\mathbb{R}$. An arithmetic $\mathbb{K}$-divisor $\overline{\mathscr{H}}$ is said to be *adequate* if there exist ample arithmetic divisors $\overline{\mathscr{H}}_1,\dots\overline{\mathscr{H}}_l$, $a_1,\dots,a_l\in\mathbb{K}_{>0}$, and a nonnegative continuous function $f\in C^0_\infty(X)$ such that

$$\overline{\mathscr{H}}=a_1\overline{\mathscr{H}}_1+\cdots+a_l\overline{\mathscr{H}}_l+(0,f).$$

Proof. (1): Note that, if $\mathrm{F}^t(X, n\overline{E}) = \mathrm{H}^0(X, nE)$ for a $t > 0$ and for every $n \gg 1$, then we have $\mathrm{vol}^0(\overline{E}) = \mathrm{vol}(E)$.

Step 1. Let $\overline{\mathscr{H}}$ be an adequate arithmetic $\mathbb{R}$-divisor on a normal model $\mathscr{X}$ as in Remark 3.7. Since $\overline{\mathscr{H}}$ can be approximated by adequate arithmetic $\mathbb{Q}$-divisors on $\mathscr{X}$, we have $\mathrm{vol}^0(\overline{D}) = \mathrm{vol}(D)$ by using the above note and the continuity of vol^0 (Proposition 3.4 (2)).

Step 2. Let $\overline{\mathscr{D}} = (\mathscr{D}, g_{\overline{\mathscr{D}}})$ be a nef and big arithmetic $\mathbb{R}$-divisor on a normal model $\mathscr{X}$: that is, $\mathscr{D}$ is relatively nef, $g_{\overline{\mathscr{D}}}$ is plurisubharmonic, and $h_{\overline{\mathscr{D}}}(x) \geqslant 0$ for every $x \in X(\overline{K})$. By [10, Proposition 6.2.2 (2)], $\overline{\mathscr{D}} + \overline{\mathscr{H}}$ is adequate for every adequate arithmetic $\mathbb{R}$-divisor $\overline{\mathscr{H}}$. Thus by the continuity and Step 1, we have $\mathrm{vol}^0(\overline{\mathscr{D}}^{\mathrm{a}}) = \mathrm{vol}(\mathscr{D} \cap X)$.

Step 3. Let $\overline{D} = \left(D, \sum_{v \in \Sigma} g_v[v]\right)$ be a nef and big adelic $\mathbb{R}$-divisor on X, $(\mathscr{X}_U, \mathscr{D}_U)$ a model of definition for $\overline{D}$, and $S := \mathrm{Spec}(O_K) \backslash U$. By Definition 2.3, for $\varepsilon > 0$, we can find an O_K-model $(\mathscr{X}, \mathscr{D}')$ of (X, D) such that $\mathscr{X} \times_{\mathrm{Spec}(O_K)} U \cong \mathscr{X}_U$, $\mathscr{D}' \cap \mathscr{X}_U = \mathscr{D}_U$, $\mathscr{D}'$ is relatively nef, and $\|g_P^{(\mathscr{X}, \mathscr{D}')} - g_P\|_{\sup} < \varepsilon$ for every $P \in S$. Let ϖ_P be a uniformizer of O_{K_P}, and let

$$\mathscr{D} := \mathscr{D}' + \varepsilon \sum_{P \in S} \frac{\mathscr{X}_P}{-\log |\varpi_P|_P^2}.$$

Then $\overline{D} \leqslant (\mathscr{D}, g_\infty)^{\mathrm{a}} \leqslant \overline{D} + 2\varepsilon \sum_{P \in S}[P]$, and $(\mathscr{D}, g_\infty)$ is nef and big. Since $\mathrm{vol}^0((\mathscr{D}, g_\infty)^{\mathrm{a}}) = \mathrm{vol}(D)$, we have $\mathrm{vol}^0(\overline{D}) = \mathrm{vol}(D)$ by the continuity. $\square$

Lemma 3.8. *If D is big, then the following are equivalent.*

(a) *$\overline{D}$ is big.*
(b) *$G_{\overline{D}}(u) > 0$ for a $u \in \Delta(D)$.*
(c) *$\mathrm{vol}^s(\overline{D}) > 0$ for an $s > 0$.*
(d) *$e_{\max}^{\mathrm{asy}}(\overline{D}) > 0$.*

Proof. (a) $\Rightarrow$ (b) is clear by (3.6).

(b) $\Rightarrow$ (c): Fix a $t_0 \in \mathbb{R}$ such that $t_0 < e_{\max}^{\mathrm{asy}}(\overline{D})$. If $t_0 \geqslant 0$, then $\mathrm{vol}^0(\overline{D}) \geqslant \mathrm{vol}^{t_0}(\overline{D}) > 0$ by [2, Lemma 1.6]. Thus we assume that $t_0 < 0$ and that there exists a $u_0 \in \Delta^{t_0}(\overline{D})$ such that $s_0 := G_{\overline{D}}(u_0) > 0$. We take an s with $0 < s < s_0$. For $u \in \Delta^{t_0}(\overline{D})$ and for $p \in \mathbb{R}$ with $0 \leqslant p \leqslant (s_0 - s)/(s_0 - t_0) \leqslant 1$, we have

$$G_{\overline{D}}((1-p)u_0 + pu) \geqslant (1-p)s_0 + pt_0 \geqslant s.$$

Hence (b) implies that $\mathrm{vol}^s(\overline{D}) > 0$.

(c) $\Rightarrow$ (d): Since $\mathrm{F}^{ns}(X, n\overline{D}) \neq \{0\}$ for all $n \gg 1$, we have $e_{\max}^{\mathrm{asy}}(\overline{D}) \geqslant s > 0$.

(d) $\Rightarrow$ (a): Let $\overline{H} = \left(H, \sum_{v \in \Sigma} h_v[v]\right)$ be a big adelic $\mathbb{R}$-divisor. Since D is big, $\mathrm{H}^0(X, nD - H) \neq \{0\}$ for every $n \gg 1$. Thus by Lemma 2.3, one can find nonzero sections $\phi \in \mathrm{H}_f^0(X, n\overline{D} - \overline{H})$ and $\psi \in \mathrm{H}_f^0(X, n\overline{D})$ such that $\|\psi\|_{\sup}^{ng_\infty} < 1$ for an $n \gg 1$. Since

$$\|\psi^m \cdot \phi\|_{\sup}^{(m+1)ng_\infty - h_\infty} \leqslant (\|\psi\|_{\sup}^{ng_\infty})^m \cdot \|\phi\|_{\sup}^{ng_\infty - h_\infty},$$

$\widehat{\mathrm{H}}^0(X, m\overline{D} - \overline{A}) \neq \{0\}$ for an $m \gg 1$. Thus $\widehat{\mathrm{vol}}(\overline{D}) \geqslant \widehat{\mathrm{vol}}(\overline{H}) > 0$ and $\overline{D}$ is big. $\square$

Lemma 3.9. *Suppose that D is big.*

(1) *The following are equivalent.*
  (a) *$\overline{D}$ is peudo-effective.*

(b) $\Delta^0(\overline{D}) \neq \emptyset$.
(c) $\sup_{u\in\Delta(D)} G_{\overline{D}}(u) \geqslant 0$.
(d) $\overline{D} + (0, \varepsilon[\infty])$ *is big for every* $\varepsilon > 0$.

(2) *If* $\mathrm{vol}^0(\overline{D}) > 0$*, then* $\overline{D}$ *is pseudo-effective.*

Remark 3.10. Since $G_{\overline{D}}$ is upper semicontinuous and $\Delta(D)$ is compact, $G_{\overline{D}}$ attains a maximum in $\Delta(D)$.

Proof. (1) (a) $\Rightarrow$ (b): Fix a big adelic $\mathbb{R}$-divisor $\overline{A}$ with $\overline{A} \geqslant 0$. Then we have

$$\Delta^0(\overline{D}) = \bigcap_{\varepsilon>0} \Delta^0(\overline{D} + \varepsilon\overline{A})$$

and $\Delta^0(\overline{D}) \neq \emptyset$. (b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (d): Since $G_{\overline{D}+(0,2\varepsilon[\infty])}(u) = G_{\overline{D}}(u) + \varepsilon$, we have

$$\sup_{u\in\Delta(D)} G_{\overline{D}+(0,2\varepsilon[\infty])}(u) > 0$$

for every $\varepsilon > 0$. Thus by Lemma 3.8, we conclude.

(d) $\Rightarrow$ (a): Since $\widehat{\mathrm{vol}}(\overline{D} + \varepsilon[\infty] + \overline{A}) \geqslant \widehat{\mathrm{vol}}(\overline{A})$ for $\varepsilon > 0$, we have $\widehat{\mathrm{vol}}(\overline{D} + \overline{A}) > 0$ for every big adelic $\mathbb{R}$-divisor $\overline{A}$. Thus $\overline{D}$ is pseudo-effective.

(2) is clear by (1). $\square$

4. Arithmetic σ-invariants

Let X be a smooth projective variety of dimension d over K, and let $\xi \in X$ be a point (not necessarily closed) on X. We fix an embedding $K \subset \overline{K}$, and let $\xi_1, \dots, \xi_r \in X_{\overline{K}}$ be the points underlying ξ, that is, $\xi_i \mapsto \xi$. Then the multiplicity of D at $\xi \in X$ satisfies

$$\mathrm{mult}_\xi(D) := \mathrm{mult}_{\xi_1}(D_{\overline{K}}) = \cdots = \mathrm{mult}_{\xi_r}(D_{\overline{K}}). \tag{4.1}$$

Given a *big* adelic $\mathbb{R}$-divisor $\overline{D}$, we define the *arithmetic* σ*-invariant* of $\overline{D}$ at $\xi \in X$ as

$$\widehat{\sigma}_\xi(\overline{D}) := \inf\left\{\mathrm{mult}_\xi(E) \;\middle|\; \overline{E} \sim_{\mathbb{R}} \overline{D} \text{ and } \overline{E} \geqslant 0\right\} \geqslant 0.$$

Lemma 4.1. *Let* $\overline{D}$ *be a big adelic* $\mathbb{R}$*-divisor.*

(1) *For any* $\phi \in \mathrm{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R}$*, we have* $\widehat{\sigma}_\xi(\overline{D} + \widehat{(\phi)}) = \widehat{\sigma}_\xi(\overline{D})$.
(2) *For any* $t \in \mathbb{R}_{\geqslant 0}$*, we have* $\widehat{\sigma}_\xi(t\overline{D}) = t\,\widehat{\sigma}_\xi(\overline{D})$.
(3) *If* $\overline{E}$ *is big, then* $\widehat{\sigma}_\xi(\overline{D} + \overline{E}) \leqslant \widehat{\sigma}_\xi(\overline{D}) + \widehat{\sigma}_\xi(\overline{E})$*, and, if* $\overline{E}$ *is effective, then* $\widehat{\sigma}_\xi(\overline{D} + \overline{E}) \leqslant \widehat{\sigma}_\xi(\overline{D}) + \mathrm{mult}_\xi(E)$.
(4) *For* $\overline{E}_1, \dots, \overline{E}_m \in \widehat{\mathrm{Div}}^{\mathrm{a}}_{\mathbb{R}}(X)$*,* $v_1, \dots, v_n \in \Sigma$*, and* $\varphi_j \in C^0_{v_j}(X)$*, we have*

$$\lim_{\substack{\varepsilon_1,\dots,\varepsilon_m\to 0\\ \|\varphi_1\|_{\sup},\dots,\|\varphi_n\|_{\sup}\to 0}} \widehat{\sigma}_\xi\left(\overline{D} + \sum_{i=1}^m \varepsilon_i\overline{E}_i + \sum_{j=1}^n (0, \varphi_j[v_j])\right) = \widehat{\sigma}_\xi(\overline{D}).$$

(5) *If* $\overline{D}$ *is nef and big, then* $\widehat{\sigma}_\xi(\overline{D}) = 0$.
(6) *If* $\overline{D}$ *is big, then*

$$\begin{aligned}\widehat{\sigma}_\xi(\overline{D}) &= \inf\left\{\mathrm{mult}_\xi(E) \;\middle|\; \overline{E} \sim_{\mathbb{R}} \overline{D} \text{ and } \overline{E} \geqslant 0\right\}\\ &= \inf\left\{\mathrm{mult}_\xi(E) \;\middle|\; \overline{E} \sim_{\mathbb{Q}} \overline{D} \text{ and } \overline{E} \geqslant 0\right\}.\end{aligned}$$

(7) *The function* $X \ni \xi \mapsto \widehat{\sigma}_\xi(\overline{D}) \in \mathbb{R}_{\geqslant 0}$ *is upper semicontinuous.*

(8) *Let x be a closed point, let $\pi : Y \to X$ be the blowing-up with center $\{x\}$, and let Γ_x be the exceptional divisor. Then we have $\widehat{\sigma}_x(\overline{D}) = \widehat{\sigma}_{\Gamma_x}(\pi^*\overline{D})$.*

Proof. One can find proofs of (1) – (6) in [11, Proposition 7.2.3]. For the reader's convenience, we include a proof of (4) (see [11, Claim 7.2.1.1]).

(4): This formally follows from (2), (3). In fact, the function $\widehat{\sigma}_\xi : \widehat{\mathrm{Big}}^{\mathrm{a}}_{\mathbb{R}}(X) \to \mathbb{R}$ is convex by (2), (3). Since

$$\begin{aligned}\widehat{\sigma}_\xi&\left(\overline{D} + \sum_{i=1}^m \varepsilon_i \overline{E}_i + \sum_{j=1}^n \|\varphi_j\|_{\sup}(0, [v_j])\right)\\ &\leqslant \widehat{\sigma}_\xi\left(\overline{D} + \sum_{i=1}^m \varepsilon_i \overline{E}_i + \sum_{j=1}^n (0, \varphi_j[v_j])\right)\\ &\leqslant \widehat{\sigma}_\xi\left(\overline{D} + \sum_{i=1}^m \varepsilon_i \overline{E}_i - \sum_{j=1}^n \|\varphi_j\|_{\sup}(0, [v_j])\right)\end{aligned}$$

by (3), we may assume that $\varphi_j \equiv 0$ for all j. Moreover, since each $\overline{E}_i$ is a difference of two big adelic $\mathbb{R}$-divisors, we may assume that $\overline{E}_1, \dots, \overline{E}_m$ are all big. Since

$$\begin{aligned}\widehat{\sigma}_\xi\left(\overline{D} + \sum_{i=1}^m |\varepsilon_i| \overline{E}_i\right) + \sum_{i=1}^m (|\varepsilon_i| - \varepsilon_i)\, \widehat{\sigma}_\xi(\overline{E}_i) &\leqslant \widehat{\sigma}_\xi\left(\overline{D} + \sum_{i=1}^m \varepsilon_i \overline{E}_i\right)\\ &\leqslant \widehat{\sigma}_\xi\left(\overline{D} - \sum_{i=1}^m |\varepsilon_i| \overline{E}_i\right) + \sum_{i=1}^m (|\varepsilon_i| + \varepsilon_i)\, \widehat{\sigma}_\xi(\overline{E}_i)\end{aligned}$$

by (2), (3), we may assume that $\varepsilon_1, \dots, \varepsilon_m$ are of the same sign.

By (2), (3), we have

$$\widehat{\sigma}_\xi\left(\overline{D} + \sum_{i=1}^m \varepsilon_i \overline{E}_i\right) \leqslant \widehat{\sigma}_\xi(\overline{D}) + \sum_{i=1}^m \varepsilon_i\, \widehat{\sigma}_\xi(\overline{E}_i) \tag{4.2}$$

and

$$\widehat{\sigma}_\xi(\overline{D}) \leqslant \widehat{\sigma}_\xi\left(\overline{D} - \sum_{i=1}^m \varepsilon_i \overline{E}_i\right) + \sum_{i=1}^m \varepsilon_i\, \widehat{\sigma}_\xi(\overline{E}_i) \tag{4.3}$$

for sufficiently small $\varepsilon_1 \geqslant 0, \dots, \varepsilon_m \geqslant 0$. Hence

$$\limsup_{\varepsilon_1,\dots,\varepsilon_m \to 0+} \widehat{\sigma}_\xi\left(\overline{D} + \sum_{i=1}^m \varepsilon_i \overline{E}_i\right) \leqslant \widehat{\sigma}_\xi(\overline{D}) \quad \text{and} \quad \liminf_{\varepsilon_1,\dots,\varepsilon_m \to 0+} \widehat{\sigma}_\xi\left(\overline{D} - \sum_{i=1}^m \varepsilon_i \overline{E}_i\right) \geqslant \widehat{\sigma}_\xi(\overline{D}).$$

Since $\overline{D}$ is big, we can fix $\delta_1, \dots, \delta_m \in \mathbb{R}_{>0}$ such that $\overline{D} - \delta_1\overline{E}_1, \dots, \overline{D} - \delta_m\overline{E}_m$ are still all big. By (2), (3), we have

$$(1 + \varepsilon_1 + \cdots + \varepsilon_m)\, \widehat{\sigma}_\xi(\overline{D}) \leqslant \widehat{\sigma}_\xi\left(\overline{D} + \sum_{i=1}^m \varepsilon_i \delta_i \overline{E}_i\right) + \sum_{i=1}^m \varepsilon_i\, \widehat{\sigma}_\xi(\overline{D} - \delta_i \overline{E}_i) \tag{4.4}$$

and

$$\widehat{\sigma}_\xi\left(\overline{D} - \sum_{i=1}^m \varepsilon_i \delta_i \overline{E}_i\right) \leqslant (1 - \varepsilon_1 - \cdots - \varepsilon_m)\, \widehat{\sigma}_\xi(\overline{D}) + \sum_{i=1}^m \varepsilon_i\, \widehat{\sigma}_\xi(\overline{D} - \delta_i \overline{E}_i) \tag{4.5}$$

for sufficiently small $\varepsilon_1 \geqslant 0, \dots, \varepsilon_m \geqslant 0$. Hence

$$\liminf_{\varepsilon_1,\dots,\varepsilon_m \to 0+} \widehat{\sigma}_\xi\left(\overline{D} + \sum_{i=1}^m \varepsilon_i \overline{E}_i\right) \geqslant \widehat{\sigma}_\xi(\overline{D}) \quad \text{and} \quad \limsup_{\varepsilon_1,\dots,\varepsilon_m \to 0+} \widehat{\sigma}_\xi\left(\overline{D} - \sum_{i=1}^m \varepsilon_i \overline{E}_i\right) \leqslant \widehat{\sigma}_\xi(\overline{D}).$$

(7): Since

$$\left\{ \xi \in X \mid \widehat{\sigma}_\xi(\overline{D}) < u \right\} = \bigcup_{\substack{\overline{E} \sim_{\mathbb{R}} \overline{D}, \\ \overline{E} \geqslant 0}} \{ \xi \in X \mid \mathrm{mult}_\xi(E) < u \}$$

for $u \in \mathbb{R}$, the assertion follows from Lemma 3.1 (1).

(8): This follows from [6, Example 4.3.9]. □

Lemma 4.2. *Let Γ be a prime divisor on X. Fix an adelic divisor $\overline{\mathscr{H}}^{\mathrm{a}}$ associated to an ample arithmetic divisor $\overline{\mathscr{H}}$ of C^∞-type on a normal model $\mathscr{X}$ of X. Then, for any effective adelic $\mathbb{R}$-divisor $\overline{D}$, we have*

$$\mathrm{mult}_\Gamma(D) \leqslant \frac{\widehat{\deg}\left(\overline{\mathscr{H}}^{\mathrm{a}\cdot d} \cdot \overline{D}\right)}{\widehat{\deg}\left((\overline{\mathscr{H}}^{\mathrm{a}}|_\Gamma)^{\cdot d}\right)}.$$

Proof. There exists a finite set $S \subset \Sigma_f$ such that, given any $\varepsilon > 0$, one can find an effective arithmetic $\mathbb{R}$-divisor $\overline{\mathscr{D}}$ on a normal model $\mathscr{X}'$ of X such that

- there exists a morphism $\mu : \mathscr{X}' \to \mathscr{X}$ that extends the identity,
- the closure $\widetilde{\Gamma}$ of Γ in $\mathscr{X}'$ is Cartier, and
- $0 \leqslant \overline{\mathscr{D}}^{\mathrm{a}} - \overline{D} \leqslant \varepsilon\left(0, \sum_{P \in S}[P]\right)$.

We set $a := \mathrm{mult}_\Gamma(D)$. Since $\mathscr{D} - a\widetilde{\Gamma}$ is still an effective Cartier divisor, we can decompose $\overline{\mathscr{D}}$ as

$$\overline{\mathscr{D}} = a(\widetilde{\Gamma}, g) + \overline{\mathscr{E}},$$

where g is a nonnegative Γ-Green function and $\overline{\mathscr{E}}$ is an effective arithmetic $\mathbb{R}$-divisor (see [10, Proposition 2.4.2]). Then we have

$$\begin{aligned}
\widehat{\deg}\left(\overline{\mathscr{H}}^{\mathrm{a}\cdot d} \cdot \overline{D}\right) + \varepsilon\,\widehat{\deg}\left(\overline{\mathscr{H}}^{\mathrm{a}\cdot d} \cdot \left(0, \sum_{P \in S}[P]\right)\right) &\geqslant \widehat{\deg}\left(\mu^*\overline{\mathscr{H}}^{\cdot d} \cdot \overline{\mathscr{D}}\right) \\
&\geqslant a\,\widehat{\deg}\left(\mu^*\overline{\mathscr{H}}^{\cdot d} \cdot (\widetilde{\Gamma}, g)\right) \\
&\geqslant a\,\widehat{\deg}\left((\overline{\mathscr{H}}^{\mathrm{a}}|_\Gamma)^{\cdot d}\right)
\end{aligned}$$

for every $\varepsilon > 0$. Hence we conclude the proof. □

Proposition 4.3. *Let $\overline{D}$ be a pseudo-effective adelic $\mathbb{R}$-divisor and let $\overline{A}$ be a nef and big adelic $\mathbb{R}$-divisor. Then the limit*

$$\lim_{\varepsilon \to 0+} \widehat{\sigma}_\xi(\overline{D} + \varepsilon\overline{A})$$

exists in $\mathbb{R}_{\geqslant 0}$ and does not depend on $\overline{A}$.

Proof. Since $\widehat{\sigma}_\xi(\overline{A}) = 0$, the function

$$\mathbb{R}_{>0} \ni \varepsilon \mapsto \widehat{\sigma}_\xi(\overline{D} + \varepsilon\overline{A}) \in \mathbb{R}_{\geqslant 0}$$

is monotone decreasing by Lemma 4.1 (3). By taking a closed point in $\overline{\{\xi\}}$, we may assume that ξ is a smooth closed point x in X. Let $\pi : Y \to X$ be the blowing-up

with center $\{x\}$, and let Γ_x be the exceptional divisor. Then, by choosing adelic divisor $\overline{\mathscr{H}}^{\mathrm{a}}$ on Y as in Lemma 4.2, we have

$$\widehat{\sigma}_x(\overline{D}+\varepsilon\overline{A})=\widehat{\sigma}_{\Gamma_x}(\pi^*(\overline{D}+\varepsilon\overline{A}))\leqslant\frac{\widehat{\deg}\left(\overline{\mathscr{H}}^{\mathrm{a}\cdot d}\cdot\pi^*(\overline{D}+\varepsilon\overline{A})\right)}{\widehat{\deg}\left((\overline{\mathscr{H}}^{\mathrm{a}}|_{\Gamma_x})^{\cdot d}\right)}$$

for every $\varepsilon>0$. Thus we have $\widehat{\sigma}_x(\overline{D})<+\infty$.

Let $\overline{A}'$ be another nef and big adelic $\mathbb{R}$-divisor. Since $\overline{A}$ is big, there exists a $\delta>0$ such that $\overline{A}-\delta\overline{A}'$ is big. By Lemma 4.1 (3),

$$\widehat{\sigma}_\xi(\overline{D}+\varepsilon\overline{A})\leqslant\widehat{\sigma}_\xi(\overline{D}+\varepsilon\delta\overline{A}')+\varepsilon\,\widehat{\sigma}_\xi(\overline{A}-\delta\overline{A}')$$

for every $\varepsilon>0$. Thus $\lim_{\varepsilon\to 0+}\widehat{\sigma}_\xi(\overline{D}+\varepsilon\overline{A})\leqslant\lim_{\varepsilon\to 0+}\widehat{\sigma}_\xi(\overline{D}+\varepsilon\overline{A}')$. By changing $\overline{A}$ for $\overline{A}'$, we conclude the proof. □

Definition 4.1. Fix a nef and big adelic $\mathbb{R}$-divisor $\overline{A}$. For a pseudo-effective adelic $\mathbb{R}$-divisor $\overline{D}$, we define

$$\widehat{\sigma}_\xi(\overline{D}):=\lim_{\varepsilon\to 0+}\widehat{\sigma}_\xi(\overline{D}+\varepsilon\overline{A}),$$

which does not depend on the choice of $\overline{A}$ by Proposition 4.3. We set

$$\widehat{\mathrm{NBs}}(\overline{D}):=\{\xi\in X\mid\widehat{\sigma}_\xi(\overline{D})>0\}.$$

Remark 4.4.
(1) If $\overline{D}$ is not big, then our $\widehat{\sigma}_\xi(\overline{D})$(Definition 4.1) is different from the $\mu_{\mathbb{K},\xi}(\overline{D})$ ($\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{Q}$) given by Moriwaki ([9], [10], [11]). We have $\widehat{\sigma}_\xi(\overline{D})\leqslant\mu_{\mathbb{R},\xi}(\overline{D})\leqslant\mu_{\mathbb{Q},\xi}(\overline{D})$ in general, and the two equalities hold if $\overline{D}$ is big. In [12], one can find an example where $\overline{D}$ is nef but $\mu_{\mathbb{R},\xi}(\overline{D})=+\infty$.
(2) We can define $\sigma_\xi(D)$ and $\mathrm{NBs}(D)$ for a pseudo-effective $\mathbb{R}$-divisor D in the same way as above (see [13, Chap. III, §1] for details).

We extend Lemma 4.1 to the pseudo-effective case as follows.

Proposition 4.5. *Let $\overline{D}$ be a pseudo-effective adelic $\mathbb{R}$-divisor.*

(1) *For any $\phi\in\mathrm{Rat}(X)^\times\otimes_{\mathbb{Z}}\mathbb{R}$, we have $\widehat{\sigma}_\xi(\overline{D}+\widehat{(\phi)})=\widehat{\sigma}_\xi(\overline{D})$.*
(2) *For any $t\in\mathbb{R}_{\geqslant 0}$, we have $\widehat{\sigma}_\xi(t\overline{D})=t\,\widehat{\sigma}_\xi(\overline{D})$.*
(3) *If $\overline{E}$ is also pseudo-effective, then $\widehat{\sigma}_\xi(\overline{D}+\overline{E})\leqslant\widehat{\sigma}_\xi(\overline{D})+\widehat{\sigma}_\xi(\overline{E})$, and, if $\overline{E}$ is effective, then $\widehat{\sigma}_\xi(\overline{D}+\overline{E})\leqslant\widehat{\sigma}_\xi(\overline{D})+\mathrm{mult}_\xi(E)$.*
(4) *Let $\overline{E}_1,\dots,\overline{E}_m\in\widehat{\mathrm{Div}}^{\mathrm{a}}_{\mathbb{R}}(X)$ and $v_1,\dots,v_n\in\Sigma$. Let $(\varepsilon_{1k})_{k=1}^\infty,\dots,(\varepsilon_{mk})_{k=1}^\infty$ be sequences of real numbers that converge to zero, and let $(\varphi_{1k})_{k=1}^\infty\subset C^0_{v_1}(X),\dots,(\varphi_{nk})_{k=1}^\infty\subset C^0_{v_n}(X)$ be sequences that uniformly converge to zero. If $\overline{D}+\sum_{i=1}^m\varepsilon_{ik}\overline{E}_i+\sum_{j=1}^n(0,\varphi_{jk}[v_j])$ is pseudo-effective for every k, we have*

$$\liminf_{k\to\infty}\widehat{\sigma}_\xi\left(\overline{D}+\sum_{i=1}^m\varepsilon_{ik}\overline{E}_i+\sum_{j=1}^n(0,\varphi_{jk}[v_j])\right)\geqslant\widehat{\sigma}_\xi(\overline{D}).$$

(5) *For $\overline{E}_1,\dots,\overline{E}_m\in\widehat{\mathrm{PE}}^{\mathrm{a}}_{\mathbb{R}}(X)$, $v_1,\dots,v_n\in\Sigma$, and $\varphi_j\in C^0_{v_j}(X)$ with $\varphi_j\geqslant 0$, we have*

$$\lim_{\substack{\varepsilon_1,\dots,\varepsilon_m\to 0+\\ \varphi_1\geqslant 0,\dots,\varphi_n\geqslant 0\\ \|\varphi_1\|_{\sup},\dots,\|\varphi_n\|_{\sup}\to 0}}\widehat{\sigma}_\xi\left(\overline{D}+\sum_{i=1}^m\varepsilon_i\overline{E}_i+\sum_{j=1}^n(0,\varphi_j[v_j])\right)=\widehat{\sigma}_\xi(\overline{D}).$$

(6) *If $\overline{D}$ is nef, then $\widehat{\sigma}_\xi(\overline{D}) = 0$.*
(7) *For $\xi \in X$, there exists an intersection of countably many nonempty Zariski open subsets, $U \subset \overline{\{\xi\}}$, such that, for every $x \in U$, $\widehat{\sigma}_x(\overline{D}) = \widehat{\sigma}_\xi(\overline{D})$. In particular,*

$$\widehat{\mathrm{NBs}}(\overline{D}) = \{x \in X \mid x \text{ is closed and } \widehat{\sigma}_x(\overline{D}) > 0\}.$$

(8) *For any nef and big adelic $\mathbb{R}$-divisor $\overline{A}$, we have*

$$\widehat{\mathrm{NBs}}(\overline{D}) = \bigcup_{t>0} \widehat{\mathrm{NBs}}(\overline{D} + t\overline{A}).$$

Remark 4.6. As we shall see in Example 5.2, Proposition 4.5 (4) does not hold without the conditions that $\varepsilon_i \geqslant 0$ for every i and $\varphi_j \geqslant 0$ for every j. In particular, the function $\widehat{\sigma}_\xi : \widehat{\mathrm{PE}}^{\mathrm{a}}_{\mathbb{R}}(X) \to \mathbb{R}$ is not continuous (but still is lower semicontinuous) over the boundary of $\widehat{\mathrm{PE}}^{\mathrm{a}}_{\mathbb{R}}(X)$.

Proof. (1) – (3), (6) follow from Lemma 4.1 (1) – (3), (5), respectively.

(4): We fix a nef and big adelic $\mathbb{R}$-divisor $\overline{A}$. Given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\widehat{\sigma}_\xi(\overline{D} + \delta\overline{A}) \geqslant \widehat{\sigma}_\xi(\overline{D}) - \varepsilon. \tag{4.6}$$

By Lemma 4.1 (4), there exists an $N \geqslant 1$ such that

$$\left|\widehat{\sigma}_\xi\left(\overline{D} + \delta\overline{A} + \sum_{i=1}^m \varepsilon_{ik}\overline{E}_i + \sum_{j=1}^n (0, \varphi_{jk}[v_j])\right) - \widehat{\sigma}_\xi\left(\overline{D} + \delta\overline{A}\right)\right| \leqslant \varepsilon \tag{4.7}$$

for every $k \geqslant N$. Moreover, we have

$$\widehat{\sigma}_\xi\left(\overline{D} + \sum_{i=1}^m \varepsilon_{ik}\overline{E}_i + \sum_{j=1}^n (0, \varphi_{jk}[v_j])\right) \geqslant \widehat{\sigma}_\xi\left(\overline{D} + \delta\overline{A} + \sum_{i=1}^m \varepsilon_{ik}\overline{E}_i + \sum_{j=1}^n (0, \varphi_{jk}[v_j])\right) \tag{4.8}$$

since $\overline{A}$ is nef. Thus, by (4.6), (4.7), (4.8), we have

$$\widehat{\sigma}_\xi\left(\overline{D} + \sum_{i=1}^m \varepsilon_{ik}\overline{E}_i + \sum_{j=1}^n (0, \varphi_{jk}[v_j])\right) \geqslant \widehat{\sigma}_\xi(\overline{D}) - 2\varepsilon$$

for every $i \geqslant N$. Hence we conclude.

(5): Since $\overline{E}_1, \dots, \overline{E}_m$ are pseudo-effective and $(0, \varphi_1[v_1]), \dots, (0, \varphi_n[v_n])$ are effective, we have

$$\widehat{\sigma}_\xi\left(\overline{D} + \sum_{i=1}^m \varepsilon_{ik}\overline{E}_i + \sum_{j=1}^n (0, \varphi_{jk}[v_j])\right) \leqslant \widehat{\sigma}_\xi(\overline{D}) + \sum_{i=1}^m \varepsilon_i\, \widehat{\sigma}_\xi(\overline{E}_i) \tag{4.9}$$

by (2), (3). On the other hand, we have

$$\liminf_{\substack{\varepsilon_1,\dots,\varepsilon_m \to 0+ \\ \varphi_1 \geqslant 0,\dots,\varphi_n \geqslant 0 \\ \|\varphi_1\|_{\sup},\dots,\|\varphi_n\|_{\sup} \to 0}} \widehat{\sigma}_\xi\left(\overline{D} + \sum_{i=1}^m \varepsilon_i\overline{E}_i + \sum_{j=1}^n (0, \varphi_j[v_j])\right) \geqslant \widehat{\sigma}_\xi(\overline{D}) \tag{4.10}$$

by (4). Thus we conclude.

(7): Fix a nef and big adelic $\mathbb{R}$-divisor $\overline{A}$. By Lemma 4.1 (7), there exists a nonempty Zariski open subset $U_n \subset \overline{\{\xi\}}$ such that

$$\widehat{\sigma}_x\left(\overline{D}+\frac{1}{n}\overline{A}\right)=\widehat{\sigma}_\xi\left(\overline{D}+\frac{1}{n}\overline{A}\right)$$

for every $x \in U_n$. We set $U := \bigcap_{n\geqslant 1} U_n$. Then $\widehat{\sigma}_x(\overline{D}) = \widehat{\sigma}_\xi(\overline{D})$ for every $x \in U$.

(8): By (3), the inclusion $\supset$ is clear. Suppose that $\widehat{\sigma}_x(\overline{D}) > 0$. Then there exists a $t > 0$ such that $\widehat{\sigma}_x(\overline{D} + t\overline{A}) > 0$. Thus $x \in \widehat{\mathrm{NBs}}(\overline{D} + t\overline{A})$. □

To prove the main results, we recall a result of Moriwaki (Theorem 4.8) relating the arithmetic Okounkov body with the arithmetic σ-invariants (see [9, Lemma 3.3], [9, Theorem 3.4], [11, Theorem 7.3.3]). We use the same notation as in §3. Given a monomial order $\leqslant$ on $\mathbb{R}^d$ and a linear form $\ell : \mathbb{R}^d \to \mathbb{R}$, we define a new monomial order $\leqslant_\ell$ as

$$I \leqslant_\ell J \quad \Leftrightarrow \quad \begin{array}{l}\ell(I) < \ell(J), \quad \text{or} \\ \ell(I) = \ell(J) \quad \text{and} \quad I \leqslant J\end{array}$$

for $I, J \in \mathbb{R}^d$, and replace $\leqslant$ with $\leqslant_\ell$ in the construction of $\Delta(D)$ and $\Delta^t(\overline{D})$.

Lemma 4.7. *Suppose that D is big. Let $\xi \in X$ be a point such that $\overline{\{\xi\}}$ is defined by $z_1 = \cdots = z_h = 0$ at x, and let $\ell(u_1, \ldots, u_d) := u_1 + \cdots + u_h$. The Okounkov bodies below are with respect to the monomial order $\leqslant_\ell$.*

(1) *We have*

$$\sigma_\xi(D) - \mathrm{mult}_\xi(D) = \inf\{\ell(u) \mid u \in \Delta(D)\}.$$

(2) *Suppose that $\overline{D}$ is pseudo-effective. Then*

$$\widehat{\sigma}_\xi(\overline{D}) - \mathrm{mult}_\xi(D) = \inf\{\ell(u) \mid u \in \Delta^0(\overline{D}) \neq \emptyset\}.$$

Proof. We give a proof of (2). If $\overline{D}$ is big, then the assertion is nothing but [9, Lemma 3.3], [11, §7.3]. Note that $\Delta^0(\overline{D}) \neq \emptyset$ and $\overline{D} + (0, 2t[\infty])$ is big for all $t > 0$ by Lemma 3.9 (1), and that the function

$$\mathbb{R}_{>0} \ni t \mapsto \inf\{\ell(u) \mid u \in \Delta^0(\overline{D} + (0, 2t[\infty]))\} \in \mathbb{R}$$

is monotone decreasing and

$$\lim_{t\to 0+} \inf\{\ell(u) \mid u \in \Delta^0(\overline{D} + (0, 2t[\infty]))\} = \inf\{\ell(u) \mid u \in \Delta^0(\overline{D})\}$$

by Remark 3.2 (3) and Proposition 3.4 (3). Hence we have

$$\begin{aligned}\inf\{\ell(u) \mid u \in \Delta^0(\overline{D})\} &= \lim_{t\to 0+} \inf\{\ell(u) \mid u \in \Delta^0(\overline{D} + (0, 2t[\infty]))\} \\ &= \lim_{t\to 0+} \widehat{\sigma}_\xi(\overline{D} + (0, 2t[\infty])) \\ &= \widehat{\sigma}_\xi(\overline{D})\end{aligned}$$

by Proposition 4.5 (5). □

Theorem 4.8. *Suppose that D is big.*

(1) *If $\mathrm{vol}^0(\overline{D}) = \mathrm{vol}(D)$, then $\overline{D}$ is pseudo-effective and $\widehat{\sigma}_\xi(\overline{D}) = \sigma_\xi(D)$ for every $\xi \in X$. In particular, we have $\widehat{\mathrm{NBs}}(\overline{D}) = \mathrm{NBs}(D)$.*

(2) *If D is nef and $\mathrm{vol}^0(\overline{D}) = \mathrm{vol}(D)$, then $\widehat{\mathrm{NBs}}(\overline{D}) = \emptyset$.*

Proof. (1): By Proposition 3.4 (3), the condition $\mathrm{vol}^0(\overline{D}) = \mathrm{vol}(D)$ implies $\Delta^0(\overline{D}) = \Delta(D)$. Hence by Lemma 4.7 (2), we have

$$\begin{aligned}\widehat{\sigma}_\xi(\overline{D}) &= \inf\{u_1 + \cdots + u_h \,|\, (u_1, \dots, u_d) \in \Delta^0(\overline{D})\} \\ &= \inf\{u_1 + \cdots + u_h \,|\, (u_1, \dots, u_d) \in \Delta(D)\} = \sigma_\xi(D).\end{aligned}$$

(2) follows from (1) since $\mathrm{NBs}(D) = \emptyset$ for nef D ([13, Chap. III, Proposition 1.14]). □

5. Absolute minima

In this section, we prove that $\widehat{\sigma}_x(\overline{D}) = 0$ implies $h_{\overline{D}}(x) \geqslant 0$ for a pseudo-effective adelic $\mathbb{R}$-divisor $\overline{D}$ (Corollary 5.5). The outline of the proof is as follows: we fix an effective adelic $\mathbb{R}$-divisor $\overline{D}' \sim_{\mathbb{R}} \overline{D} + \overline{A}$ with small multiplicity at x and choose a suitable curve X' passing through x such that $\mathrm{mult}_x(\overline{D}' \cap X') = \mathrm{mult}_x(\overline{D}')$. Then, for any $\varepsilon > 0$, $h_{\overline{D}' \cap X'}(x) \geqslant -\varepsilon$ implies $h_{\overline{D}'}(x) \geqslant -\varepsilon$. As an application, we prove a characterization of nef adelic $\mathbb{R}$-divisors in terms of arithmetic numerical base loci (Theorem 5.6). We also evaluate the absolute minima of generically big and vertically nef adelic $\mathbb{R}$-divisors in terms of arithmetic numerical base loci (Corollary 5.7).

Proposition 5.1. *Let X be a smooth projective variety over K, let $x \in X(\overline{K})$ be a rational point, and let $\mathcal{I}$ be the ideal sheaf defining the simple point $\{x\}$ in X. Fix an adelic divisor $\overline{A}$ on X such that*

$$\left\langle \mathrm{H}^0(X, \mathcal{O}_X(A) \otimes \mathcal{I}) \cap \widehat{\mathrm{H}}^0(X, \overline{A}) \right\rangle_K \otimes_K \mathcal{O}_X \to \mathcal{O}_X(A) \otimes \mathcal{I}$$

is surjective. Then, for any effective adelic $\mathbb{R}$-divisor $\overline{D}$, we have

$$h_{\overline{D}}(x) + \mathrm{mult}_x(D) \cdot h_{\overline{A}}(x) \geqslant 0.$$

Remark 5.2. Let $\mathscr{X}$ be an O_K-model of X, and let $\mathcal{J}$ be the ideal sheaf defining $\overline{\{x\}}$ in $\mathscr{X}$. We can find an ample divisor $\mathscr{A}$ on $\mathscr{X}$ such that

$$\mathrm{H}^0(\mathscr{X}, \mathcal{O}_{\mathscr{X}}(\mathscr{A}) \otimes \mathcal{J}) \otimes_{\mathbb{Z}} \mathcal{O}_{\mathscr{X}} \to \mathcal{O}_{\mathscr{X}}(\mathscr{A}) \otimes \mathcal{J}$$

is surjective, and choose a continuous Hermitian metric on $\mathscr{A}$ such that $\mathrm{H}^0(\mathscr{X}, \mathcal{O}_{\mathscr{X}}(\mathscr{A}) \otimes \mathcal{J})$ is generated by sections s with $\|s\|_{\sup}^{\overline{\mathscr{A}}} \leqslant 1$. Then $\overline{A} := \overline{\mathscr{A}}^{\mathrm{a}}$ satisfies the conditions of Proposition 5.1.

Proof. We set $\delta := \mathrm{mult}_x(D)$. If $\delta = 0$, then x is not contained in $\mathrm{Supp}(D)$ and we have $h_{\overline{D}}(x) \geqslant 0$ by Lemma 2.4 (3). We assume $\delta > 0$ and prove the assertion by induction on dimension.

Claim 5.3. *The assertion holds if* $\dim X$ *is one.*

Proof. There exists a $\phi \in \mathrm{H}^0(X, A - x)$ such that $\overline{A} + \widehat{(\phi)}$ is effective and

$$\mathrm{mult}_x\left(D + \delta(A + (\phi))\right) = 0. \tag{5.1}$$

Since $\overline{D} + \delta(\overline{A} + \widehat{(\phi)})$ is effective and x is not contained in $\mathrm{Supp}(D + \delta(A + (\phi)))$, we have

$$h_{\overline{D}}(x) + \delta h_{\overline{A}}(x) = h_{\overline{D} + \delta(\overline{A} + \widehat{(\phi)})}(x) \geqslant 0$$

by Lemma 2.4 (3). □

Since X is smooth, we can write $D = a_1D_1 + \cdots + a_lD_l$ with effective prime Cartier divisors $D_1, \dots, D_l$ and positive real numbers $a_1, \dots, a_l \in \mathbb{R}$. Then $\mathrm{Supp}(D) := \bigcup_{i=1}^{l} \mathrm{Supp}(D_i)$.

Claim 5.4. *One can find a smooth closed subvariety $X' \subset X$ of dimension $\dim X - 1$ having the following properties.*

- *X' passes through x.*
- *X' is not contained in $\mathrm{Supp}(D) \cup \mathrm{Supp}(A)$ (in particular, one can consider the restrictions $\overline{D}|_{X'} = a_1\overline{D}_1|_{X'} + \cdots + a_l\overline{D}_l|_{X'}$ and $\overline{A}|_{X'}$).*
- *$\mathrm{mult}_x(D|_{X'}) = \mathrm{mult}_x(D)$.*

Proof. Let $r := [K(x) : \mathbb{Q}]$, and let $x_1, \dots, x_r$ be the points on X_∞^{an} underlying x. We can find a very ample line bundle H such that

$$\mathrm{H}^0(X, H \otimes \mathcal{I}) \otimes_{\mathbb{Q}} \mathcal{O}_X \to H \otimes_{\mathbb{Q}} \mathcal{I}/\mathcal{I}^2$$

is surjective. By Lemma 3.1 (2) and (4.1), a general section $s_{\mathbb{C}} \in \mathrm{H}^0(X, H \otimes \mathcal{I}) \otimes_{\mathbb{Q}} \mathbb{C}$ has the following properties:

- $\mathrm{div}(s_{\mathbb{C}})$ is generically smooth,
- $s_{\mathbb{C}}$ does not pass through any generic point of $D_{1,\mathbb{C}}, \dots, D_{l,\mathbb{C}}, A_{\mathbb{C}}$, and
- $\mathrm{mult}_{x_j}(D_{i,\mathbb{C}}|_{\mathrm{div}(s_{\mathbb{C}})}) = \mathrm{mult}_{x_j}(D_{i,\mathbb{C}})$ for every i, j.

Thus we can find an $s \in \mathrm{H}^0(X, H \otimes \mathcal{I})$ such that $s_{\mathbb{C}}$ has the all properties above, and set $X' := \mathrm{div}(s)$. □

The ideal sheaf $\mathcal{I}' := \mathcal{I}\mathcal{O}_{X'}$ defines the reduced induced scheme structure on $\{x\}$ in X'. Since

$$\begin{CD}
\left\langle \mathrm{H}^0(X', \mathcal{O}_{X'}(A|_{X'}) \otimes \mathcal{I}') \cap \widehat{\mathrm{H}}^0(X', \overline{A}|_{X'}) \right\rangle_K \otimes_K \mathcal{O}_{X'} @>>> \mathcal{O}_{X'}(A|_{X'}) \otimes \mathcal{I}' \\
@AAA @AAA \\
\left\langle \mathrm{H}^0(X, \mathcal{O}_X(A) \otimes \mathcal{I}) \cap \widehat{\mathrm{H}}^0(X, \overline{A}) \right\rangle_K \otimes_K \mathcal{O}_X @>>> \mathcal{O}_X(A) \otimes \mathcal{I}
\end{CD}$$

is commutative, we can apply the induction hypothesis and have

$$h_{\overline{D}}(x) + \mathrm{mult}_x(D) \cdot h_{\overline{A}}(x) = h_{\overline{D}|_{X'}}(x) + \mathrm{mult}_x(D|_{X'}) \cdot h_{\overline{A}|_{X'}}(x) \geqslant 0$$

by Lemma 2.4 (4). □

Corollary 5.5. *Let X be a smooth projective variety over K, let $\overline{D}$ be a pseudo-effective adelic $\mathbb{R}$-divisor, and $x \in X(\overline{K})$. If $\widehat{\sigma}_x(\overline{D}) = 0$, then we have $h_{\overline{D}}(x) \geqslant 0$.*

Proof. Let $\mathcal{I}$ be the ideal sheaf defining $\{x\}$ in X, and fix an adelic divisor $\overline{A}$ on X such that

$$\left\langle \mathrm{H}^0(X, \mathcal{O}_X(A) \otimes \mathcal{I}) \cap \widehat{\mathrm{H}}^0(X, \overline{A}) \right\rangle_K \otimes_K \mathcal{O}_X \to \mathcal{O}_X(A) \otimes \mathcal{I}$$

is surjective. Since $\widehat{\sigma}_x(\overline{D}) = 0$, given any $\varepsilon > 0$, we can find a $\phi \in \mathrm{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R}$ such that $\overline{D} + \varepsilon\overline{A} + \widehat{(\phi)}$ is effective and $\mathrm{mult}_x(\overline{D} + \varepsilon\overline{A} + \widehat{(\phi)}) < \varepsilon$. Since $h_{\overline{A}}(x) \geqslant 0$, we have

$$h_{\overline{D}}(x) + 2\varepsilon h_{\overline{A}}(x) \geqslant 0$$

by Proposition 5.1 and Lemma 2.4 (1). Hence we conclude the proof. □

If $\overline{D}$ is pseudo-effective, we have

$$\{x \in X(\overline{K}) \mid h_{\overline{D}}(x) < 0\} \subset \{x \in X(\overline{K}) \mid \widehat{\sigma}_x(\overline{D}) > 0\}$$

by Corollary 5.5, and, if $\overline{D}$ is nef, then both are empty. In view of Theorem 5.6, we can regard that these two sets measure the "non-nefness" of $\overline{D}$. As we shall see in Example 5.2, these two sets do not coincide if $\overline{D}$ is not nef. If $d = 1$ and $\overline{D}$ is big, then we know that the both sets are finite (Proposition 5.8).

Theorem 5.6. *Let X be a smooth projective variety over K and let $\overline{D}$ be a vertically nef adelic $\mathbb{R}$-divisor on X. Then the following three conditions are all equivalent.*

(a) *$\overline{D}$ is nef: that is, $h_{\overline{D}}(x) \geqslant 0$ for all $x \in X(\overline{K})$.*

(b) *There exists a nef adelic $\mathbb{R}$-divisor $\overline{A}$ such that*

$$\mathrm{vol}^0(\overline{D} + \varepsilon\overline{A}) = \mathrm{vol}(D + \varepsilon A) > 0$$

for every $\varepsilon > 0$.

(c) *$\overline{D}$ is pseudo-effective and $\widehat{\mathrm{NBs}}(\overline{D}) = \emptyset$.*

In particular, if D is big, then we can replace (b) *with the following.*

(b)$'$ $\mathrm{vol}^0(\overline{D}) = \mathrm{vol}(D)$.

Proof. (a) $\Rightarrow$ (b), (b)$'$ follows from Corollary 3.6 (1). (b), (b)$'$ $\Rightarrow$ (c) follows from Theorem 4.8, and (c) $\Rightarrow$ (a) follows from Corollary 5.5. ☐

Corollary 5.7. *If $\overline{D}$ is a vertically nef adelic $\mathbb{R}$-divisor, then the following two invariants all coincide.*

(a) $\widehat{\mu}_{\mathrm{abs}}(\overline{D}) := \inf_{x \in X(\overline{K})} h_{\overline{D}}(x)$.

(b) $\sup\{t \in \mathbb{R} \mid \overline{D} - (0, 2t[\infty]) \in \widehat{\mathrm{PE}}^{\mathrm{a}}_{\mathbb{R}}(X)$ *and* $\widehat{\mathrm{NBs}}(\overline{D} - (0, 2t[\infty])) = \emptyset\}$.

Moreover, if D is big, then the following also coincide with the above.

(c) $\inf_{u \in \Delta(D)} G_{\overline{D}}(u)$.

(d) $\sup\{t \in \mathbb{R} \mid \mathrm{vol}^t(\overline{D}) = \mathrm{vol}(D)\}$.

Moreover, if D is ample, then the following also coincides with the above.

(e) $e^{\mathrm{asy}}_{\mathrm{min}}(\overline{D})$.

Proof. If one of the above invariants is $-\infty$, then all are $-\infty$. Thus we may assume that all are finite. Since $\overline{D} - \big(0, 2\widehat{\mu}_{\mathrm{abs}}(\overline{D})[\infty]\big)$ is nef, we have (a) $\leqslant$ (b) and (a) $\leqslant$ (c). On the other hand, if $\overline{D} - (0, 2t[\infty])$ is pseudo-effective and $\widehat{\mathrm{NBs}}(\overline{D} - (0, 2t[\infty])) = \emptyset$ (resp. if $\mathrm{vol}^t(\overline{D}) = \mathrm{vol}^0(\overline{D} - 2t[\infty]) = \mathrm{vol}(D)$), $\overline{D} - (0, 2t[\infty])$ is nef by Theorem 5.6. Thus $h_{\overline{D}}(x) \geqslant t$ for every $x \in X(\overline{K})$ and (a) $\geqslant$ (b) (resp. (a) $\geqslant$ (c)). By Lemma 3.5, we have (c) = (d).

If $\overline{D}$ is an ample adelic $\mathbb{Q}$-divisor, then by [16, Corollary (5.7)] and [17, Theorem 1.10] we have (a) = (e). In general, since we can approximate $\overline{D}$ by ample adelic $\mathbb{Q}$-divisors, the assertion holds by Lemma 2.5 (3). ☐

Proposition 5.8. *Suppose that $\overline{D}$ is big and that $\overline{D}$ admits a Zariski decomposition $\overline{D} = \overline{P} + \overline{N}$: that is, $\overline{P}$ is nef, $\overline{E}$ is effective, and $\widehat{\mathrm{vol}}(\overline{P}) = \widehat{\mathrm{vol}}(\overline{D})$ (see* [7]*). Then we have $\widehat{\mathrm{NBs}}(\overline{D}) = \mathrm{Supp}(N)$. We consider the following four conditions.*

(a) $\mathrm{vol}^0(\overline{D}) = \mathrm{vol}(D)$.

(b) $\widehat{\mathrm{NBs}}(\overline{D}) = \emptyset$.

(c) *There exist a finite set $S \subset \Sigma$ and nonnegative functions φ_v on X_v^{an} such that* $\overline{N} = \left(0, \sum_{v\in S} \varphi_v[v]\right)$.

(d) $\inf_{x\in X(\overline{K})} h_{\overline{D}}(x) \geqslant 0$.

Then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d).

Proof. (a) $\Rightarrow$ (b) is clear by Theorem 4.8.

(b) $\Rightarrow$ (c): Since $\overline{D}$ admits a Zariski decomposition, (b) implies that $\mathrm{Supp}(N) = \widehat{\mathrm{NBs}}(\overline{D}) = \emptyset$. Thus (c) follows.

(c) $\Rightarrow$ (a): Since $\mathrm{vol}(P) = \mathrm{vol}(D)$, we have

$$\mathrm{vol}(D) = \mathrm{vol}(P) = \mathrm{vol}^0(\overline{P}) = \mathrm{vol}^0(\overline{D}).$$

(c) $\Rightarrow$ (d) holds since $h_{\overline{N}}(x) \geqslant 0$ for every $x \in X(\overline{K})$. □

Proposition 5.9. *Let $\overline{D} = (D, \sum_v g_v[v])$ be an adelic $\mathbb{R}$-divisor such that D is ample. We set*

$$Q(g_P) := \sup\left\{ g^{(\mathscr{X},\mathscr{D})} \;\middle|\; \begin{array}{l} (\mathscr{X},\mathscr{D}) \textit{ is an } O_{K_P}\textit{-model of } (X,D) \textit{ such that} \\ g^{(\mathscr{X},\mathscr{D})} \leqslant g_P \textit{ and } \mathscr{D} \textit{ is relatively nef} \end{array} \right\}$$

for $P \in \Sigma_f$, and

$$Q(g_\infty) := \sup\{\gamma \,|\, \gamma \textit{ is plurisubharmonic with } \gamma \leqslant g_\infty\}.$$

Suppose that $Q(g_P)$ is a D-Green function for every $P \in \Sigma_f$. Then $\overline{Q}(\overline{D}) := (D, \sum_v Q(g_v)[v])$ is the maximal vertically nef adelic $\mathbb{R}$-subdivisor of $\overline{D}$ such that $G_{\overline{Q}(\overline{D})} = G_{\overline{D}}$.

Proof. By the assumptions, $\overline{Q}(\overline{D})$ is a vertically nef adelic $\mathbb{R}$-subdivisor of $\overline{D}$. Let $t < e_{\max}^{\mathrm{asy}}(\overline{D})$. For every nef adelic $\mathbb{R}$-subdivisor $\overline{N}$ of $\overline{D} - (0, 2t[\infty])$, we have $\overline{N} \leqslant \overline{Q}(\overline{D}) - (0, 2t[\infty])$. Thus by the Fujita approximation [11, Theorem 5.1.6], we have $\Delta^t(\overline{Q}(\overline{D})) = \Delta^t(\overline{D})$. □

If X is a curve, then the conditions of Propositions 5.9 are satisfied (see [14], [11, Theorem 6.1.1]) and the infimum of $G_{\overline{D}}$ is given by the absolute minimum of $\overline{Q}(\overline{D})$:

$$\inf_{u\in\Delta(D)} G_{\overline{D}}(u) = \inf_{x\in X(\overline{K})} h_{\overline{Q}(\overline{D})}(x) \leqslant \inf_{x\in X(\overline{K})} h_{\overline{D}}(x).$$

The last equality does not hold in general if $\overline{D}$ is not vertically nef. We give a counterexample in Example 5.2.

Example 5.1. Let $\overline{D}$ be a toric metrized $\mathbb{R}$-divisor on a toric projective variety X as in [4, Definition 4.12]. After a toric blow up $\pi : X' \to X$, there exists a Zariski decomposition $\pi^*D = P + N$ [4, Proposition 4.10 (2)]. Let $\overline{P}$ be the vertically nef adelic $\mathbb{R}$-divisor corresponding to the roof functions [4, Definition 4.17]. Then $G_{\overline{P}} = G_{\pi^*\overline{D}}$ on $\Delta(P) = \Delta(\pi^*D)$.

Example 5.2. Let $\mathscr{X} := \mathbb{P}^d_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[X_0, \dots, X_d])$, $z_i := X_i/X_0$ and $\mathscr{D} := \{X_0 = 0\}$ as in [10, §9.4]. We take an $\boldsymbol{a} := (a_0, \dots, a_d) \in \mathbb{R}^{d+1}_{>0}$, and set

$$g_{\boldsymbol{a}} := \log\max\left\{a_0^2, a_1^2|z_1|^2, \dots, a_d^2|z_d|^2\right\}, \tag{5.2}$$

which is a plurisubharmonic $\mathscr{D}$-Green function of continuous type, and $\overline{\mathscr{D}}_{\boldsymbol{a}} := (\mathscr{D}, g_{\boldsymbol{a}})$. Note that $\overline{\mathscr{D}}_{\boldsymbol{a}}$ is invariant under the natural torus action.

We take a cutoff function $\rho : \mathbb{P}^d_{\mathbb{C}} \to \mathbb{R}_{\geqslant 0}$ near $x_0 := (1 : 0 : \cdots : 0)$ such that

$$\operatorname{Supp} \rho \Subset \{|z_1| < a_0/a_1\} \times \cdots \times \{|z_d| < a_0/a_d\}, \tag{5.3}$$

and set $\overline{\mathscr{D}}_{\boldsymbol{a},\rho} := \overline{\mathscr{D}}_{\boldsymbol{a}} + (0, \rho) = (\mathscr{D}, g_{\boldsymbol{a}} + \rho)$ and $g_{\boldsymbol{a},\rho} := g_{\boldsymbol{a}} + \rho$. By using the maximal value principle, one can easily see that the $\overline{\mathscr{D}}_{\boldsymbol{a}}$ gives the maximal element of the set $\left\{\overline{\mathscr{E}} \,\middle|\, \overline{\mathscr{E}} \text{ is vertically nef and } \overline{\mathscr{D}}_{\boldsymbol{a}} \leqslant \overline{\mathscr{E}} \leqslant \overline{\mathscr{D}}_{\boldsymbol{a},\rho}\right\}$. In the following, we see that the shape of the arithmetic Okounkov body of $\overline{\mathscr{D}}_{\boldsymbol{a},\rho}$ is determined by $\overline{\mathscr{D}}_{\boldsymbol{a}}$.

Claim 5.10. *For $n \geqslant 1$, we have $\mathrm{H}^0(\mathscr{X}, n\mathscr{D}) = \bigoplus_{I \in \mathbb{Z}^d_{\geqslant 0}, |I| \leqslant n} \mathbb{Z} z^I$.*

(1) *For $I \in \mathbb{Z}^d_{\geqslant 0}$ with $|I| \leqslant n$,*
$$\|z^I\|_{\sup}^{ng_{\boldsymbol{a},\rho}} = \frac{1}{a_0^{n-i_1-\cdots-i_d} \cdot a_1^{i_1} \cdots a_d^{i_d}}.$$

(2) *For $\phi := \sum_{I \in \mathbb{Z}^d_{\geqslant 0}, |I| \leqslant n} c_I z^I \in \mathrm{H}^0(\mathscr{X}, n\mathscr{D})$, we have*
$$\|\phi\|_{\sup}^{ng_{\boldsymbol{a},\rho}} \geqslant \sqrt{\sum_{I \in \mathbb{Z}^d_{\geqslant 0}, |I| \leqslant n} \left(\frac{c_I}{a_0^{n-i_1-\cdots-i_d} \cdot a_1^{i_1} \cdots a_d^{i_d}}\right)^2}.$$

Proof. (2): By (5.3), we have

$$\begin{aligned}
\|\phi\|_{\sup}^{ng_{\boldsymbol{a},\rho}} &\geqslant \sup_{|\zeta_i| = a_0/a_i} \left|\phi(\zeta) \exp\left(-\frac{ng_{\boldsymbol{a},\rho}(\zeta)}{2}\right)\right| = \frac{1}{a_0^n} \sup_{|\zeta_i| = a_0/a_i} \left|\sum_I c_I \zeta^I\right| \\
&\geqslant \sqrt{\int_0^1 \cdots \int_0^1 \left|\sum_I \frac{c_I}{a_0^{n-|I|} \cdot a^I} \exp\left(2\pi\sqrt{-1}(i_1 t_1 + \cdots + i_n t_n)\right)\right|^2 dt_1 \dots dt_n} \\
&= \sqrt{\sum_I \left(\frac{c_I}{a_0^{n-|I|} \cdot a^I}\right)^2}.
\end{aligned}$$

(1) follows from (2). $\square$

Set $\Delta := \{(u_0, \dots, u_n) \in \mathbb{R}^{n+1}_{\geqslant 0} \,|\, u_0 + \cdots + u_n = 1\}$. We define a function $G : \Delta \to \mathbb{R}$ by

$$G(u) := u_0 \log a_0 + \cdots + u_d \log a_d \tag{5.4}$$

and set $\Delta^t := \{u \in \Delta \,|\, G(u) \geqslant t\}$ for $t \in \mathbb{R}$. Since

$$\mathrm{F}^t(\mathbb{P}^d_{\mathbb{Q}}, n\overline{D}^{\mathrm{a}}_{\boldsymbol{a},\rho}) = \bigoplus_{I \in \mathbb{Z}^d_{\geqslant 0} \cap n\Delta^t} \mathbb{Q} z^I \tag{5.5}$$

by Claim 5.10, we can see that (Δ, G) gives the arithmetic Okounkov body of $\overline{\mathscr{D}}_{\boldsymbol{a},\rho}$ with respect to an arbitrarily fixed monomial order, and that $-G$ is the Legendre-Fenchel transform of $\max\{\log a_0, v_1 + \log a_1, \dots, v_d + \log a_d\}$: that is,

$$-G(u) = \sup\left\{\langle u, v\rangle - \max\{\log a_0, v_1 + \log a_1, \dots, v_d + \log a_d\} \,\middle|\, v \in \mathbb{R}^d\right\}$$

for $u \in \Delta$.

Claim 5.11. (1) *$\overline{\mathscr{D}}_{\boldsymbol{a},\rho}$ is nef if and only if $a_i \geqslant 1$ for every i and $\rho = 0$.*

(2) *$\overline{\mathscr{D}}_{\boldsymbol{a},\rho}$ is effective if and only if $\overline{\mathscr{D}}_{\boldsymbol{a}}$ is pseudo-effective.*

(3) *$\overline{\mathscr{D}}_{\boldsymbol{a},\rho}$ is big if and only if $a_i > 1$ for some i.*

(4) $\widehat{\mathrm{vol}}(\overline{\mathscr{D}}_{\boldsymbol{a},\rho}) = \widehat{\mathrm{vol}}(\overline{\mathscr{D}}_{\boldsymbol{a}}) = (d+1)! \int_{\Delta^0} G(u)\, du$ *and*

$$\widehat{\mathrm{vol}}^{\hat{\chi}}(\overline{\mathscr{D}}_{\boldsymbol{a},\rho}) = \widehat{\mathrm{vol}}^{\hat{\chi}}(\overline{\mathscr{D}}_{\boldsymbol{a}}) = (d+1)! \int_{\Delta} G(u)\, du = \log \prod_{i=0}^{d} a_i.$$

(5) *Suppose that* $\overline{\mathscr{D}}_{\boldsymbol{a},\rho}$ *is pseudo-effective. Then* $\widehat{\sigma}_{\xi}(\overline{\mathscr{D}}_{\boldsymbol{a},\rho}) = \widehat{\sigma}_{\xi}(\overline{\mathscr{D}}_{\boldsymbol{a}})$ *for every* $\xi \in \mathbb{P}^d_{\mathbb{Q}}$*, and*

$$\widehat{\mathrm{NBs}}(\overline{\mathscr{D}}_{\boldsymbol{a},\rho}) = \widehat{\mathrm{NBs}}(\overline{\mathscr{D}}_{\boldsymbol{a}}) = \bigcap_{a_i \geqslant 1} \{X_i = 0\}.$$

(6) *We have*

$$\inf_{x \in \mathscr{X}(\overline{\mathbb{Q}})} h_{\overline{\mathscr{D}}_{\boldsymbol{a},\rho}}(x) = \min\{\log a_0 + \rho(x_0), \log a_1, \dots, \log a_d\}.$$

Proof. (1)–(5) follow from the general theory ([2], Lemmas 3.8, 3.9, and 4.7 (2)).

For $x \in \mathscr{X}(\overline{\mathbb{Q}})$, we have

$$h_{\overline{\mathscr{D}}_{\boldsymbol{a},\rho}}(x) \begin{cases} = \log a_0 + \rho(x_0) & \text{if } x = x_0, \text{ and} \\ \geqslant \min\{\log a_1, \dots, \log a_d\} & \text{if } x \neq x_0. \end{cases} \tag{5.6}$$

Thus (6) holds. □

Suppose that $\min\{a_1, \dots, a_d\} = a_1$, $\max\{a_1, \dots, a_d\} = a_d$, $0 < a_0 < 1 \leqslant a_1$, and $\rho(x_0) \geqslant \log(a_1/a_0)$. Then

$$\inf_{x \in \mathscr{X}(\overline{\mathbb{Q}})} h_{\overline{\mathscr{D}}_{\boldsymbol{a},\rho}}(x) = \log a_1 \geqslant 0, \quad \widehat{\sigma}_{x_0}(\overline{\mathscr{D}}_{\boldsymbol{a},\rho}) = \frac{-\log a_0}{\log a_d - \log a_0} > 0,$$

and $\widehat{\mathrm{NBs}}(\overline{\mathscr{D}}_{\boldsymbol{a},\rho}) = \overline{\{x_0\}} \neq \emptyset$.

Suppose that $\overline{\mathscr{D}}_{\boldsymbol{a},\rho}$ is pseudo-effective and $0 < a_i \leqslant 1$ for every i. Then

$$\widehat{\sigma}_{\{X_i=0\}}(\overline{\mathscr{D}}_{\boldsymbol{a},\rho}) = \begin{cases} 0 & \text{if } a_i = 1, \text{ and} \\ 1 & \text{if } 0 < a_i < 1. \end{cases}$$

Given any $u = (u_0, \dots, u_d) \in \Delta^0$, we can see that $\overline{\mathscr{D}}_{\boldsymbol{a},\rho} + \widehat{(z_1^{u_1} \cdots z_d^{u_d})} \geqslant 0$. This gives a weak Zariski decomposition of $\overline{\mathscr{D}}_{\boldsymbol{a},\rho}$ in the sense that $\widehat{\mathrm{vol}}(\overline{\mathscr{D}}_{\boldsymbol{a},\rho}) = \widehat{\mathrm{vol}}(\widehat{(z_1^{-u_1} \cdots z_d^{-u_d})}) = 0$.

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